

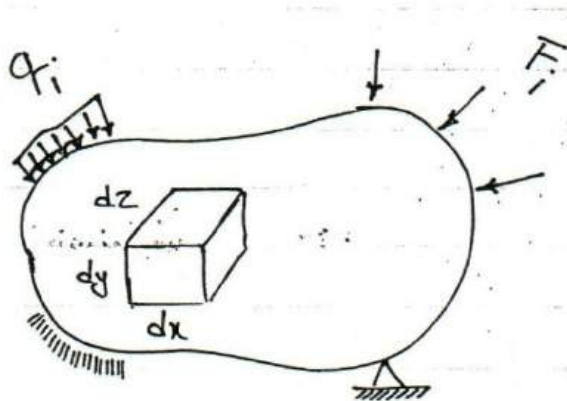
« جزوه المان محدود (مقدمه) » دکتر خوشی

جنبه اله صادره

طلبه دوم:

انرژی:

حسی در شرایط تعادل (استاتیکی) در نظر گرفته که تحت نیروها سطحی و حجمی (بدنه) قرار گرفته باشد.
برای تعیین معادله تعادل (دیفانسیل) حاکم بر جسم با استفاده از المانی داریم:



$$q_i \Rightarrow \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}$$

$$\text{Surface Loading} \Rightarrow \{q_i, F_i\} = t_i$$

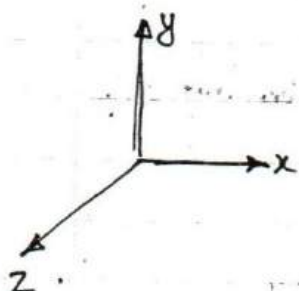
$$F_i \Rightarrow \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

$$\underline{\tilde{q}} = \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} \quad \underline{\tilde{F}} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \quad \underline{\tilde{t}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

سطح : surface : S

حجم : volume : V

body forces: $b \rightarrow \underline{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$



$$[\sigma_{ij}] = [\underline{\sigma}] = \begin{bmatrix} \sigma_{xx} & \overset{\tau}{\sigma_{xy}} & \overset{\tau}{\sigma_{xz}} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$\Rightarrow$

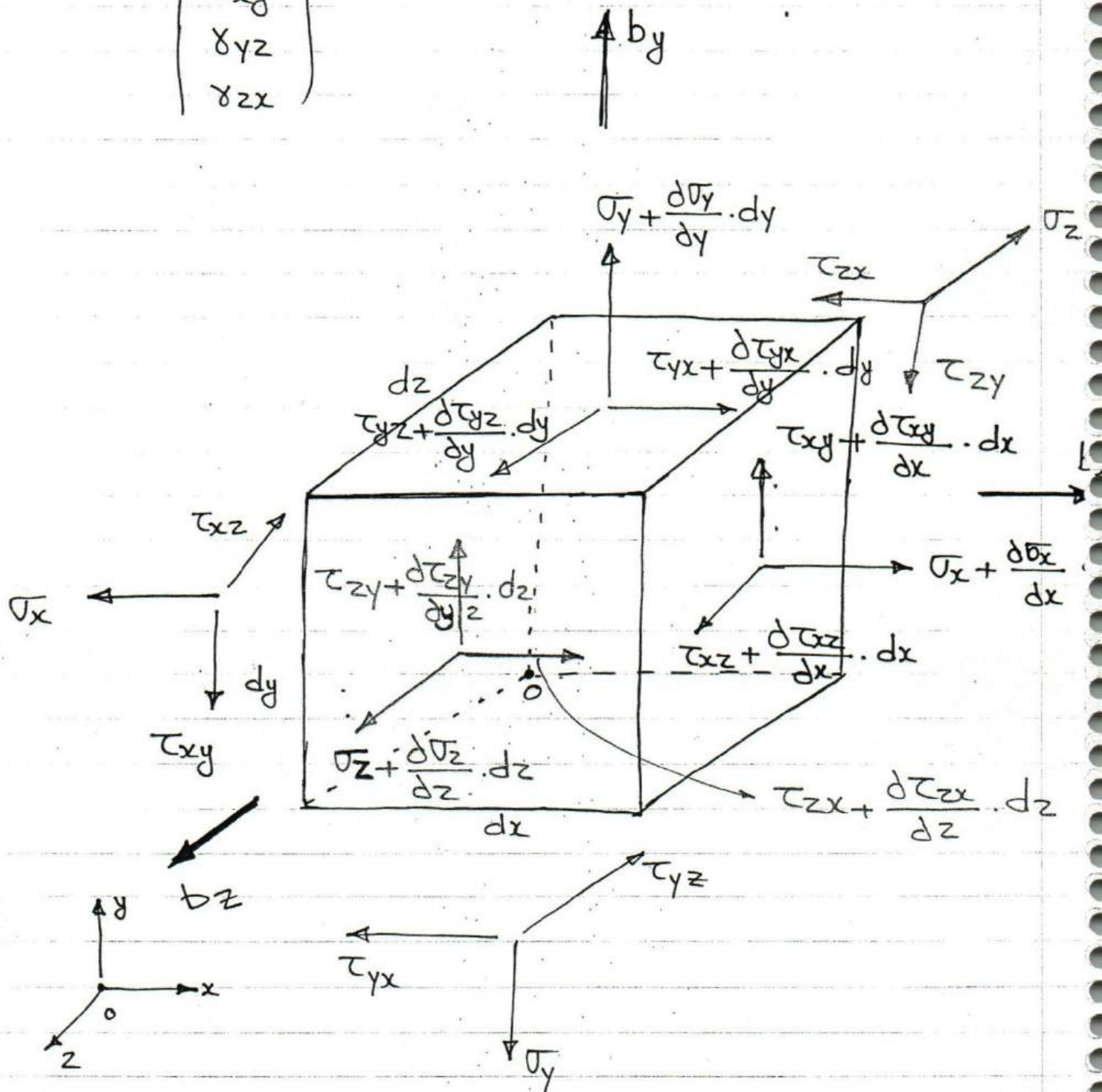
تانسور تنش

برطاری $\Rightarrow \left\{ \begin{array}{l} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{zy} \end{array} \right\} = \underline{\sigma} = \{ \sigma_{ij} \}$

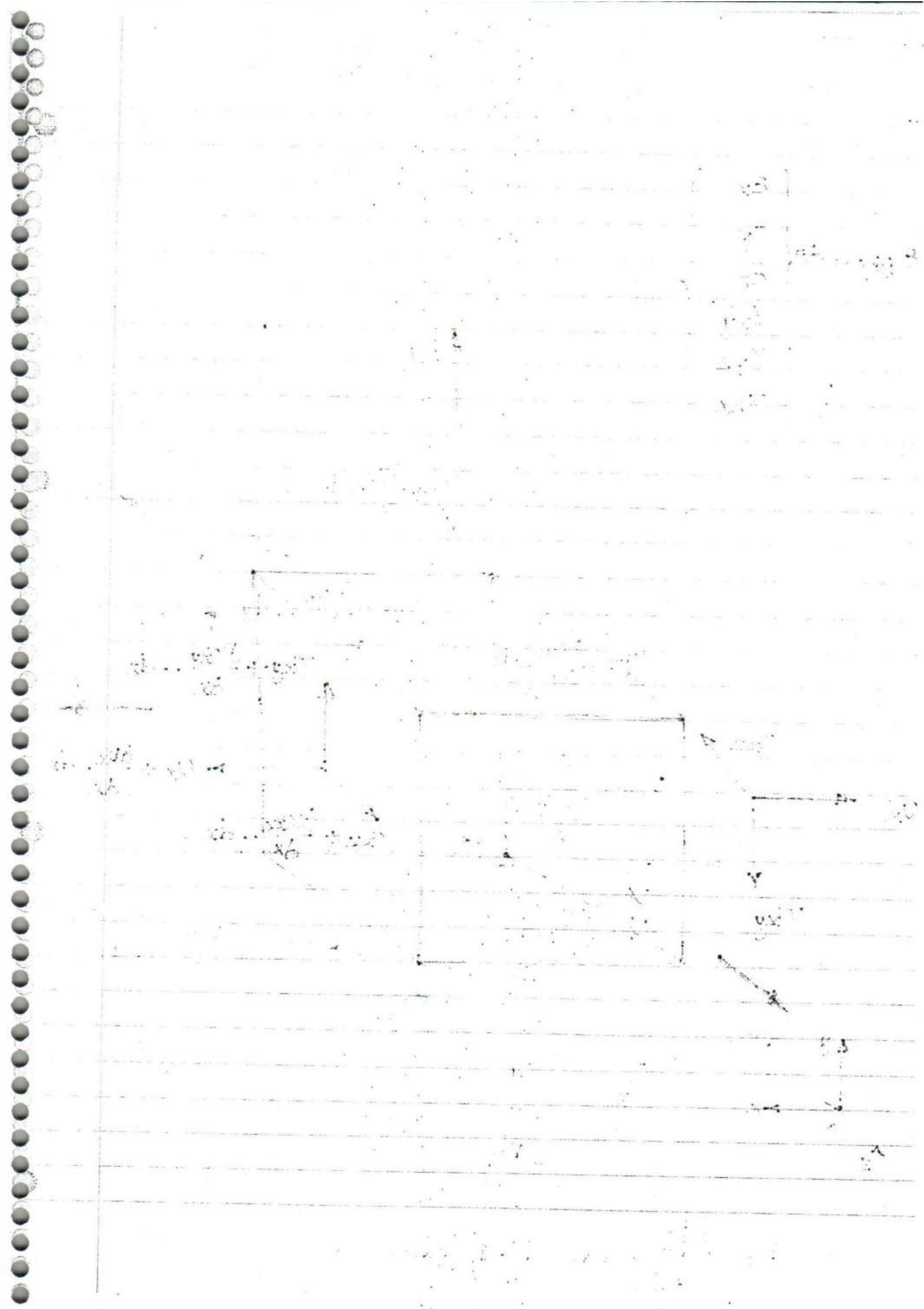
$$[\underline{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \overset{\gamma}{\epsilon_{xy}} & \overset{\gamma}{\epsilon_{xz}} \\ \gamma_{yx} & \epsilon_{yy} & \overset{\gamma}{\epsilon_{yz}} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

2.

$$\{\epsilon_{ij}\} = \underline{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$



$$b_i = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \Rightarrow \text{body force}$$



3.

$$\Sigma F_x = 0 \Rightarrow \frac{\partial \sigma_x}{\partial x} [dx \cdot (dy \cdot dz)] + \frac{\partial \tau_{yx}}{\partial y} [dy \cdot (dx \cdot dz)] + \frac{\partial \tau_{zx}}{\partial z} [dz \cdot (dx \cdot dy)] + b_x [dx \cdot dy \cdot dz] = 0$$

$$\rightarrow \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0 \quad \text{I}$$

$$\Sigma F_y = 0 \rightarrow \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0$$

$$\rightarrow \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0 \quad \text{II}$$

$$\Sigma F_z = 0 \rightarrow \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0 \quad \text{III}$$

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0 \end{cases}$$

$$\frac{\partial \bar{\sigma}_{ji}}{\partial x_j} + b_i = 0$$

$$\rightarrow i, j = x, y, z$$

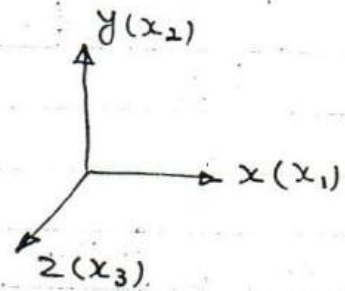
$$\downarrow \quad \downarrow \quad \downarrow$$

$$x_1 \quad x_2 \quad x_3$$

or

$$\bar{\sigma}_{ji} = \bar{\sigma}_{ij}$$

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + b_i = 0$$



معادله اول $\rightarrow i = x_1 \quad j = x_1, x_2, x_3 \rightarrow \Sigma F_x = 0$

معادله دوم $\rightarrow i = y = x_2 \quad j = x_1, x_2, x_3 \rightarrow \Sigma F_y = 0$

معادله سوم $\rightarrow i = z = x_3 \quad j = x_1, x_2, x_3 \rightarrow \Sigma F_z = 0$

4.

Virtual displacement Theorem

Virtual force Theorem

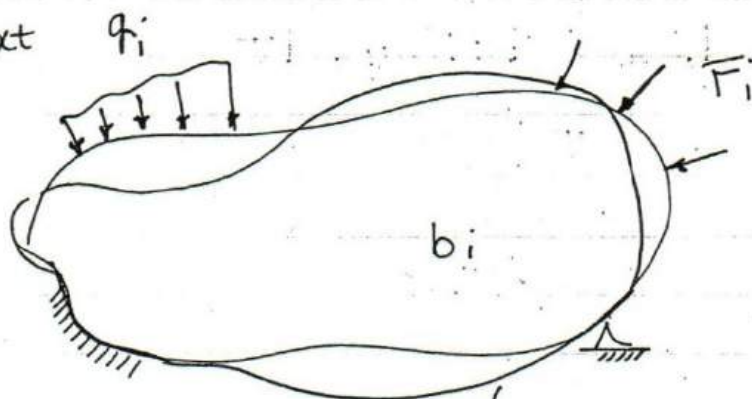
مقیه جابه جایی مجازی

مقیه کار مجازی

مقیه مجازی
شیری

مقیه جابه جایی مجازی ← برای یک سیستم در حال تعادل تحت بارگذاری ها سطحی و بدنه (حجمی)

در صورتی که یک جابه جایی دلخواه همساز با شرایط مرزی در نظر گرفته شود کاری داخلی با کار خارجی مجازی مجازی مجازی
 δW_{int}
 δW_{ext}
 مجازی یا اختیاری
 (δu_i)
 برابر است.



$i \rightarrow x, y, z$

δu_i

virtual displacement

$$\delta u = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}$$

جابه جایی

δ نشان مجازی بودن

$$\delta W_{int} = \delta W_{ext}$$



درش ها ناشی از جابه جایی است ← درش ها مجازی هستند

نشان مجازی
 $\delta \epsilon_i \rightarrow$ درش مجازی

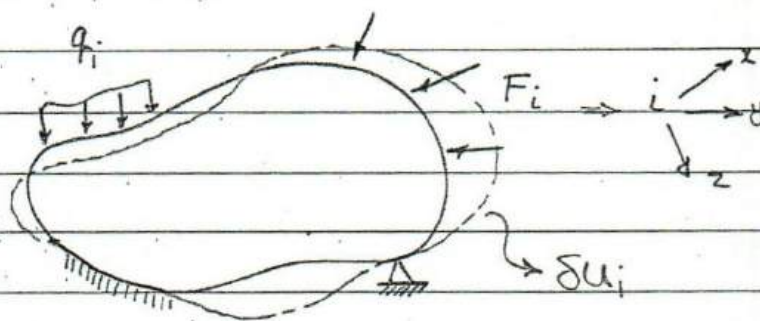
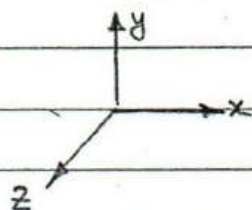
$$\delta \underline{\underline{\epsilon}} = \begin{Bmatrix} \delta \epsilon_x \\ \delta \epsilon_y \\ \delta \epsilon_z \\ \delta \gamma_{xy} \\ \delta \gamma_{yz} \\ \delta \gamma_{zx} \end{Bmatrix}$$

$$\delta W_{ext} = \int_S \underbrace{t_i}_{\text{حقیقی}} \times \underbrace{\delta u_i}_{\text{مجازی}} \cdot dS + \int_V \underbrace{b_i}_{\text{حقیقی}} \cdot \underbrace{\delta u_i}_{\text{مجازی}} \cdot dV$$

$$\delta W_{int} = \int_V \underbrace{\sigma_{ij}}_{\text{حقیقی}} \cdot \underbrace{\delta \epsilon_{ij}}_{\text{مجازی}} \cdot dV$$

$$\underline{t} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} \rightsquigarrow \delta \underline{u} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} \longleftarrow \underline{b}_i = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \rightsquigarrow \delta \underline{\underline{\epsilon}} = \begin{Bmatrix} \delta \epsilon_x \\ \delta \epsilon_y \\ \delta \epsilon_z \\ \delta \gamma_{xy} \\ \delta \gamma_{yz} \\ \delta \gamma_{zx} \end{Bmatrix}$$



$$t_i = \{F_i, q_i\}$$

t_i شامل F_i ها و q_i ها می باشد

$t_i \Rightarrow$ Surface loading

body force $\Rightarrow b_i \Rightarrow$ تنش نیروی وزن

$$\frac{\delta \sigma_{ij}}{\delta x_j} + b_i = 0$$

موضع جابه جایی مجازی $\leftarrow$ برای یک سیستم در حال تعادل که تحت جابه جایی مجازی δu_i قرار گیرد

گزینه کار مجازی داخلی با کار مجازی خارجی برابر است.

δ نشان دهنده مجازی بودن δu_i است.

$$\delta W_{ext} = \delta W_{int}$$

$$\underline{t} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

$$\underline{\delta u} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}$$

$$\underline{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

$$\delta W_{ext} = \int_S \underline{t}_i \cdot \delta u_i dS + \int_V \underline{b}_i \cdot \delta u_i dV$$

$$\delta W_{int} = \int_V \sigma_{ij} \delta \epsilon_{ij} dV$$

$$\delta W_{int} = \int_V \sigma_{ij} \delta \epsilon_{ij} dV$$

$$\underline{\underline{\epsilon}} = \begin{Bmatrix} \delta \epsilon_x \\ \delta \epsilon_y \\ \delta \epsilon_z \\ \delta \gamma_{xy} \\ \delta \gamma_{yz} \\ \delta \gamma_{zx} \end{Bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

گرنش از مشخصات جانبی است چون جانبی مجازی است بنابراین گرنش ها مجازی است

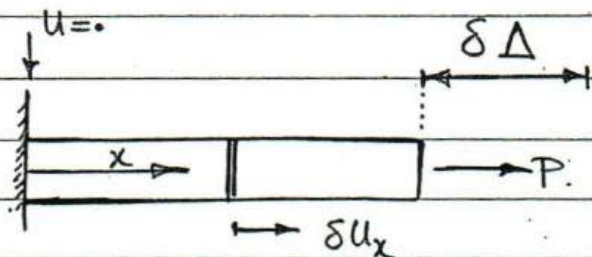
تنش ها به واسطه نیروها است چون نیروها حقیقی است تنش ها هم حقیقی اند.

میدان گرنش مجازی $\rightarrow \delta \epsilon_{ij}$

میدان جانبی مجازی $\rightarrow \delta u_i$

مثال: برای عضو محوری تحت نیروی کششی P در انتهای آن مقدار دفرمه مطلوب است بررسی قضیه جانبی

مجازی (تقابل بین) کار مجازی داخلی و خارجی (در هر یک از سه حالت زیر)



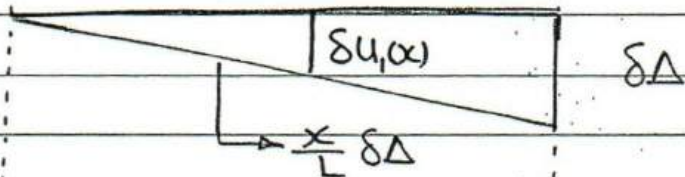
۱- میزان جانبی مجازی

۲- میزان جانبی مجازی مرتبه دوم - مرتبه های بالاتر را خودمان بررسی کنیم

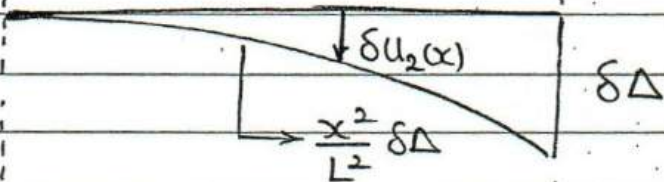
۳- میزان جانبی مجازی سینوسی



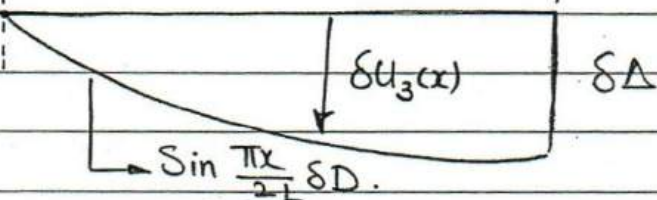
میزان جانبی مجازی خطی



میزان جانبی مجازی مرتبه دوم



میزان جانبی مجازی سینوسی



$\delta \Delta$ انتخابی است حالتی تواند متفاوت باشد یعنی $\delta \Delta_1, \delta \Delta_2, \delta \Delta_3$ اما با راسی برابر فرض کردیم

$$\delta u_1(x) = \frac{x}{L} \delta \Delta$$

$$\delta u_2(x) = \frac{x^2}{L^2} \delta \Delta$$

$$\delta u_3 = \frac{\sin \pi x / 2L}{2L} \delta \Delta$$

$$\delta W_{\text{ext}} = \int_S \cancel{q_i} \delta u_i ds + \cancel{F_i} \delta u_i + \int_V \cancel{b_i} \delta u_i dV$$

$$\delta W_{\text{ext}} = P \cdot \delta \Delta$$

$$\delta W_{\text{int}} = \int_V \sigma_{ij} \cdot \delta \epsilon_{ij} dV = \int_V \underbrace{\sigma_x}_{\frac{d}{dx}(\delta u)} \cdot \underbrace{\delta \epsilon_x}_{A \cdot dx} dV$$

$$\epsilon_x = \frac{du}{dx} \longrightarrow \delta \epsilon_x = \delta \left(\frac{du}{dx} \right) = \frac{d}{dx} (\delta u)$$

$$\delta W_{\text{int}} = \int_L P \cdot \frac{d}{dx} (\delta u(x)) dx$$

حالت اول: میان خطی :

$$\delta W_{\text{int}} = \int_0^L P \left[\frac{1}{L} \delta \Delta \right] dx = \frac{P}{L} \cdot \delta \Delta \cdot L = P \cdot \delta \Delta$$

$$\longrightarrow W_{\text{int}} = W_{\text{ext}}$$

حالت 2: میدان مجازی مرتبه دوم:

$$W_{int}^2 = \int_0^L P \left[\frac{2x}{L^2} \delta \Delta \right] dx = P \left. \frac{2x^2}{2L^2} \delta \Delta \right|_0^L = P \cdot \delta \Delta.$$

حالت 3: میدان مجازی سینوسی:

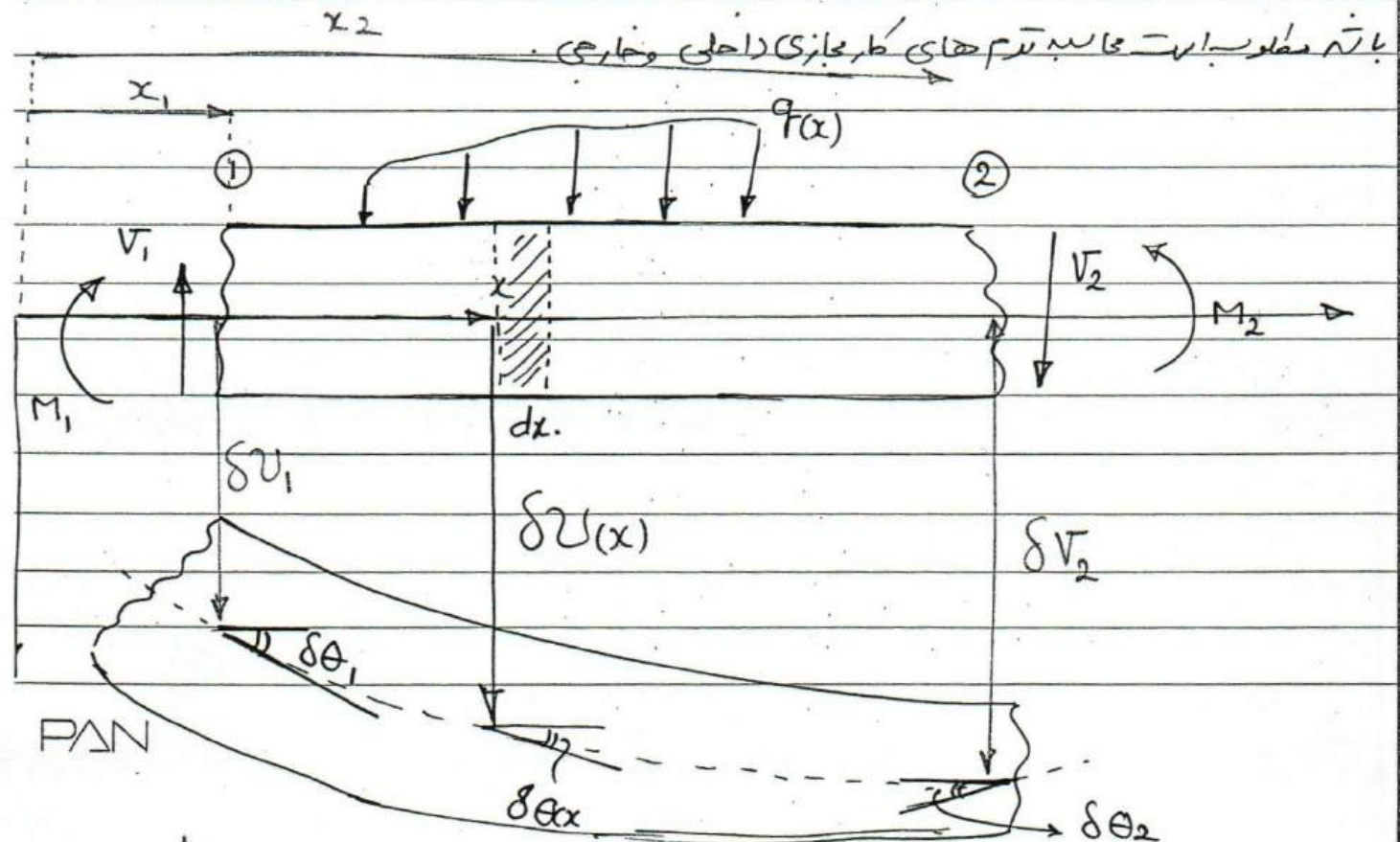
$$W_{int}^3 = \int_0^L P \left[\frac{\pi}{2L} \cos \frac{\pi x}{2L} \delta \Delta \right] dx$$

$$= P \left. \frac{\pi}{2L} \cdot \frac{2L}{\pi} \sin \frac{\pi x}{2L} \delta \Delta \right|_0^L = P \cdot \delta \Delta$$

مثال: برای عضو حقیقی شکل زیر که تحت بار عریض $q(x)$ ، برش‌ها V_1 ، V_2 و گشتاورها M_1 ، M_2

در دو انتهای آن قرار گرفته در صورتی که سیرجیت میدان جابه جایی اختیاری (مجازی) $\delta U(x)$

باشد مطلوب بار مجازی x_2 و بار مجازی داخلی و خارجی.



$$\frac{dM}{dx} = V(x).$$

$$\rightarrow \frac{d^2 M}{dx^2} = -q(x)$$

$$\frac{dV(x)}{dx} = -q(x)$$

$$\rightarrow \frac{d^2 M}{dx^2} + q(x) = 0$$

میان جانبی مجازی

$$\frac{d^2 M}{dx^2} + q(x) = 0 \xrightarrow{* \delta U(x)} \left[\frac{d^2 M}{dx^2} + q(x) \right] \delta U(x) = 0$$

$$\int_L \left[\frac{d^2 M}{dx^2} + q(x) \right] \delta U(x) dx = 0$$

$$\int_L \underbrace{\frac{d^2 M}{dx^2}}_{dv} \underbrace{\delta U(x)}_u dx + \int_L q(x) \delta U(x) dx = 0$$

انتگرال گیری جزء به جزء

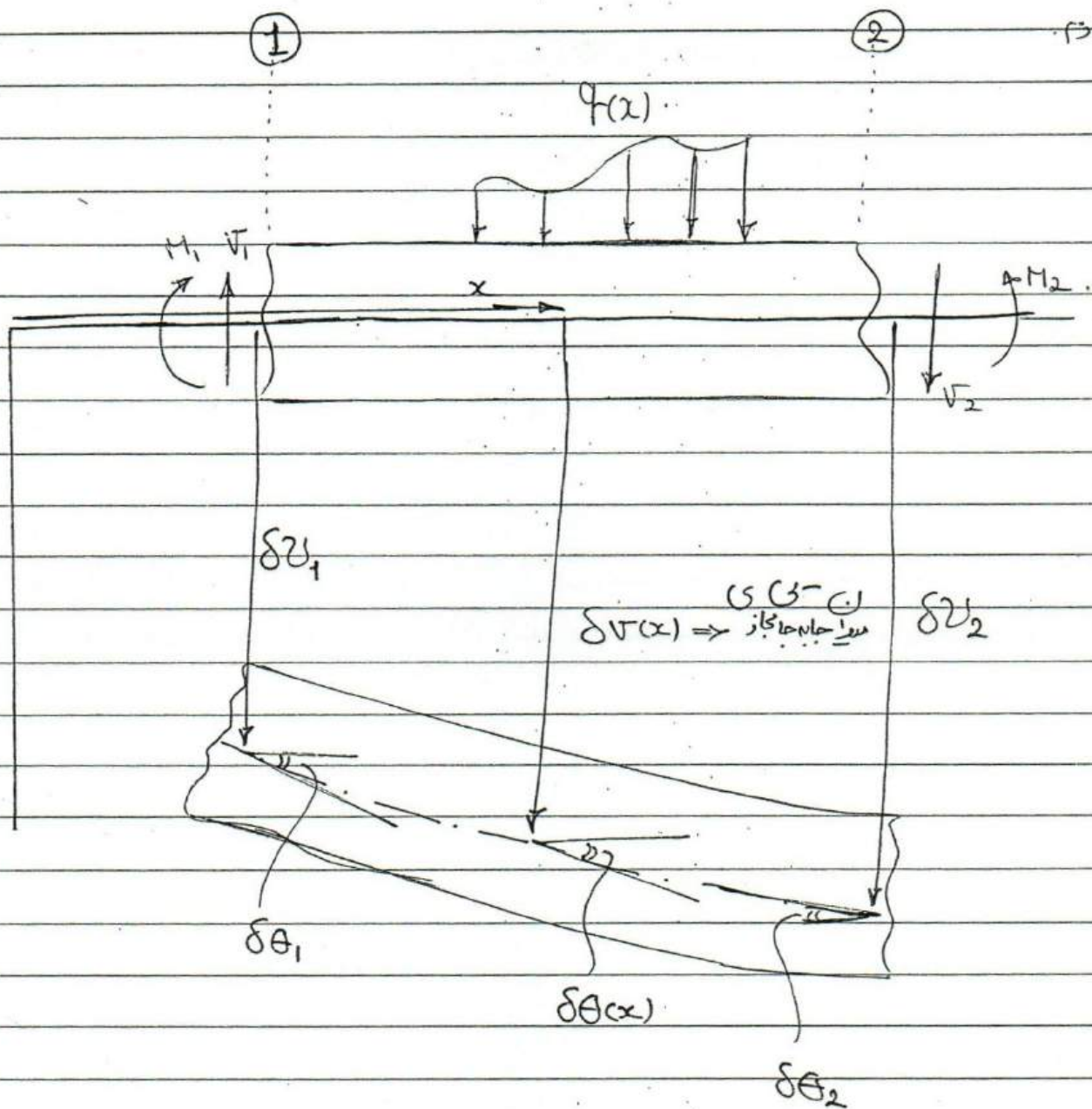
کار مجازی خارجی ناشی از $q(x)$

$$\left[\frac{dM}{dx} \delta U(x) \right]_{x_1}^{x_2} - \int_L \frac{dM}{dx} \frac{d}{dx} \left(\delta U(x) \right) dx + \int_L q(x) \delta U(x) dx = 0$$

$$(V_2 \delta V_2 - V_1 \delta V_1) - \int_L \frac{dM(x)}{dx} \delta U(x) dx + \int_L q(x) \delta U(x) dx = 0$$

کار مجازی خارجی ناشی از نیروهای برشی V_2, V_1

انتگرال گیری جزء به جزء



No.

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$$(V_2 \delta v_2 - V_1 \delta v_1) - \left[M \frac{\delta \theta}{\alpha} \right]_{x_1}^{x_2} + \int_L M(x) \frac{d}{dx} (\delta \theta) dx + \int q(x) \delta v(x) dx =$$

δK
 $\delta \left(\frac{d\theta}{dx} \right)$

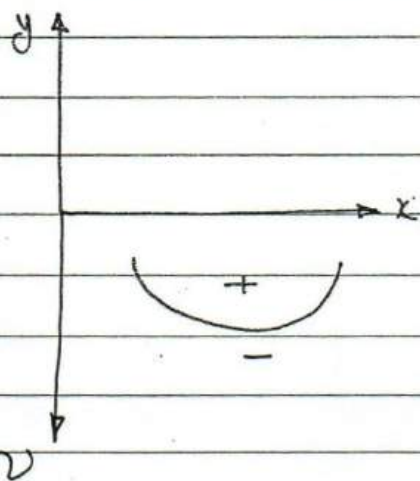
$$(V_2 \delta v_2 - V_1 \delta v_1) + (M_1 \delta \theta_1 - M_2 \delta \theta_2) - \int_L M(x) \delta K dx$$

M_2, M_1 از سرهای عضو

$$\delta \theta = \delta \left(\frac{dv}{dx} \right) = \frac{d}{dx} (\delta v)$$

$$\delta K = \frac{d}{dx} (\delta \theta)$$

$$+ \int_L q(x) \delta v(x) dx =$$



w_{ext}

$$(V_2 \delta v_2 - V_1 \delta v_1) + (M_1 \delta \theta_1 - M_2 \delta \theta_2) + \int_L q(x) \delta v(x) dx$$

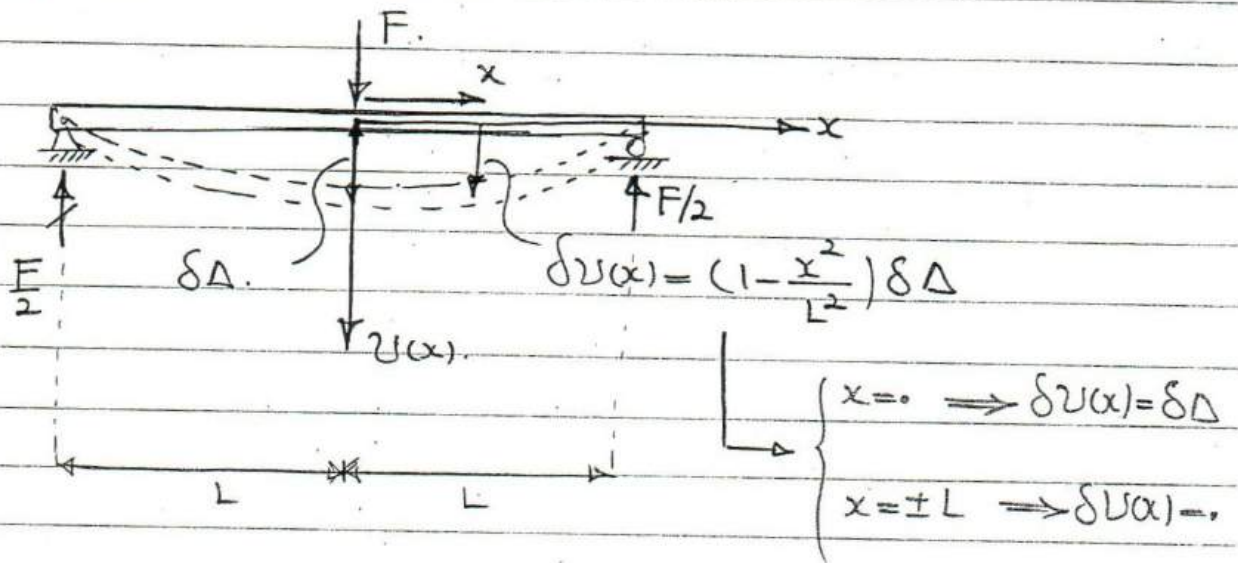
$$= \int M(x) \delta K dx$$

w_{int}

②

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$$\rightarrow w_{ext} = w_{int}$$



$$\rightarrow \delta V(x) = (1 - \frac{x^2}{L^2}) \delta \Delta$$

$$\rightarrow \delta \theta(x) = -\frac{2x}{L^2} \delta \Delta$$

$$\rightarrow \delta K(x) = -\left(\frac{d \delta \theta(x)}{dx}\right) = \frac{2}{L^2} \Delta \delta$$

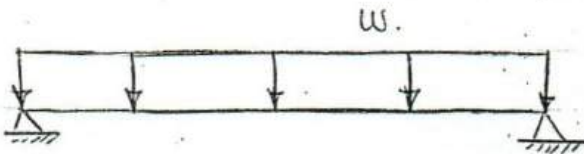
$$\begin{cases} V_1 = \frac{F}{2} \\ V_2 = -\frac{F}{2} \end{cases} \Rightarrow \begin{cases} \delta V_1 = 0 \\ \delta V_2 = 0 \end{cases}$$

$$\begin{cases} M_1 = 0 \\ M_2 = 0 \end{cases} \Rightarrow \begin{cases} \delta \theta_1 = +\frac{2}{L} \delta \Delta \rightarrow x = -L \\ \delta \theta_2 = -\frac{2}{L} \delta \Delta \rightarrow x = +L \end{cases}$$

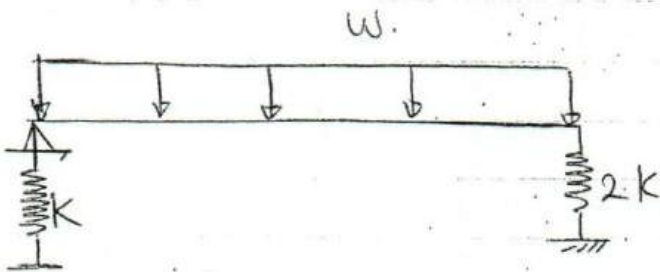
درستی قضیه کار مجازی را در سیرهای زیر بررسی کنید؟

→ تابع تغییر شکل را درجه 2 و درجه 3، سینوسی و نمایی در نظر بگیرید.

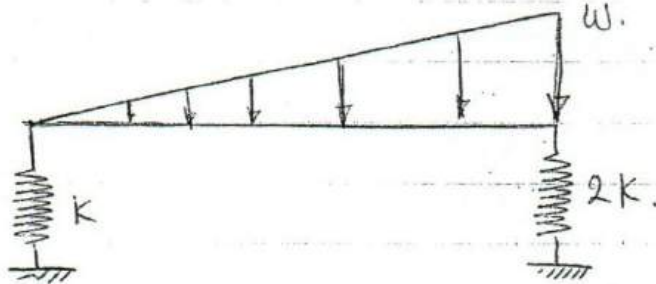
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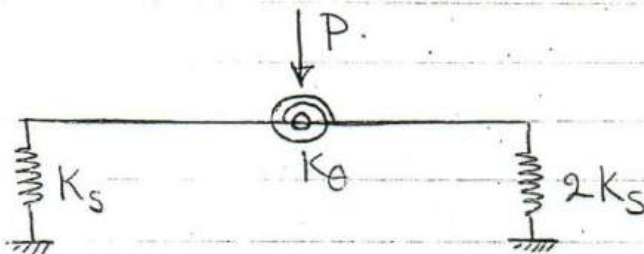
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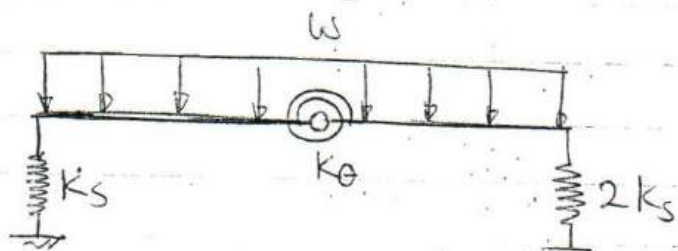
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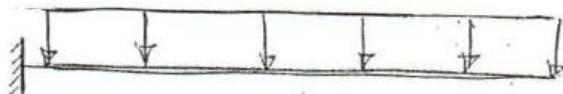
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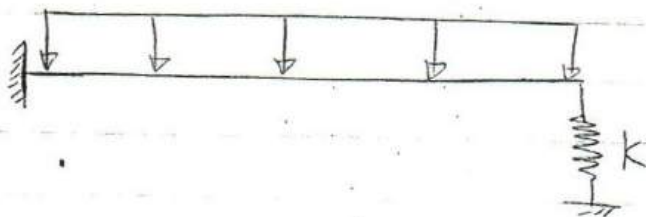
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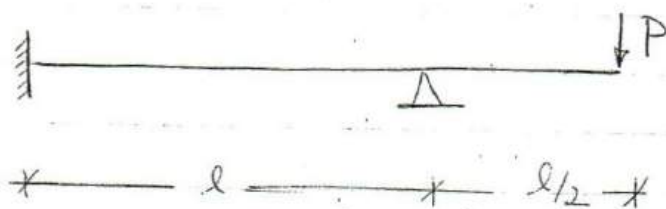
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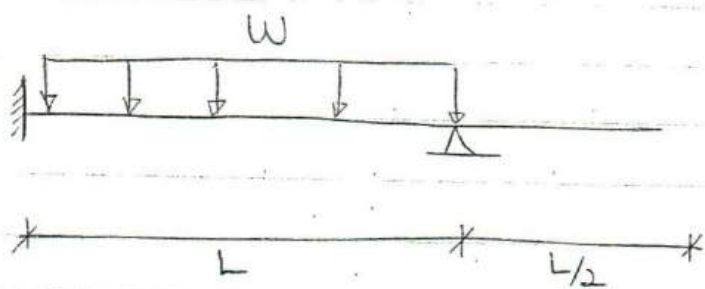
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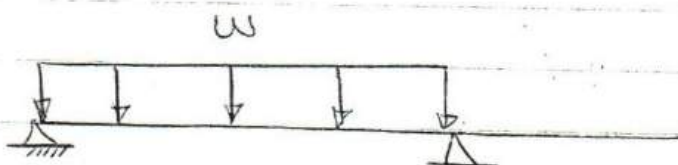
8)



9)

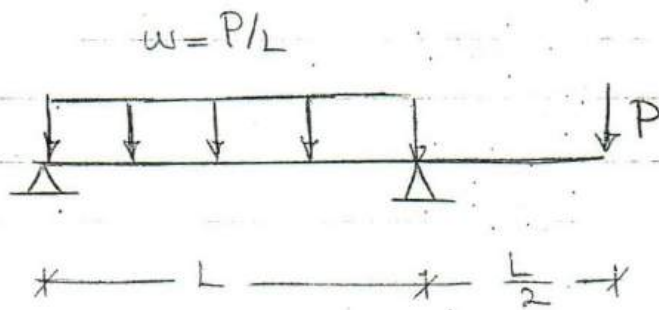


10)

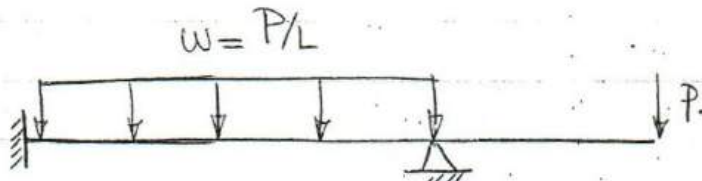


9.

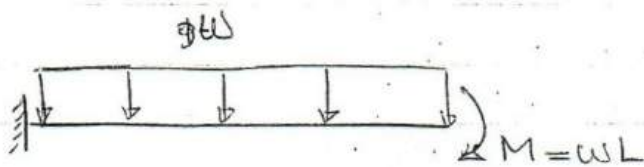
11)



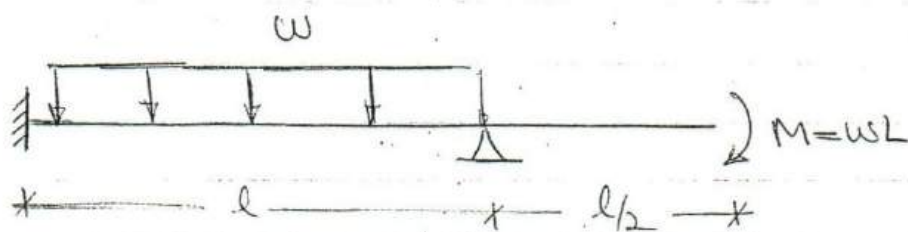
12)



13)



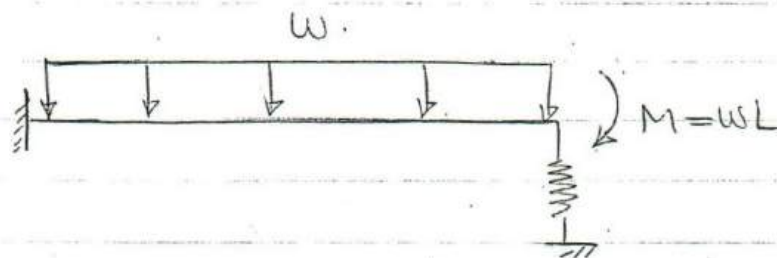
14)



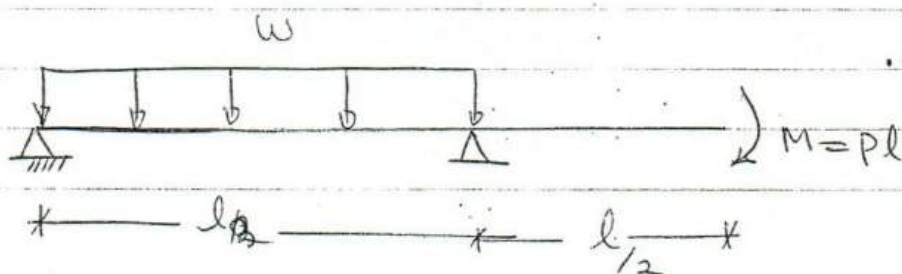
15)



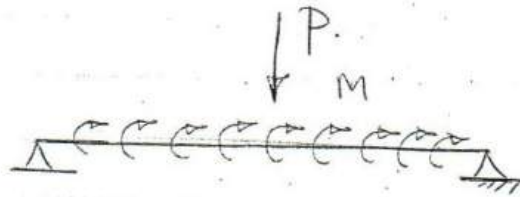
16)



17)



18)



19)



20)



21)



22)



23)

24)

$$-L \leq x \leq 0 \rightarrow M_1(x) = \frac{FL}{2} \left(1 + \frac{x}{L}\right)$$

$$0 \leq x \leq L \rightarrow M_2(x) = \frac{FL}{2} \left(1 - \frac{x}{L}\right)$$

$$(V_2 \delta V_2 - V_1 \delta V_1) + (M_1 \delta \theta_1 - M_2 \delta \theta_2) + \int_{-L}^L q(x) \delta V(x) dx = \overbrace{F \cdot \Delta \delta}$$

$$\int_{-L}^L M(x) \delta K dx = \frac{2}{L^2} \delta \Delta \left\{ \int_{-L}^0 \frac{FL}{2} \left(1 + \frac{x}{L}\right) dx + \int_0^L \left(1 - \frac{x}{L}\right) \frac{FL}{2} dx \right\}$$

از مساحت زیر نمودار دایره هم می شد حل کنیم

$$\left[x + \frac{x^2}{2L} \right]_{-L}^0 \quad \left[x - \frac{x^2}{2L} \right]_0^L$$

$$= \frac{2}{L^2} \delta \Delta \cdot \frac{FL}{2} \times 2L = F \cdot \delta \Delta$$

با تغییر شکل ها درجه ۳ و سینوسی حل شود

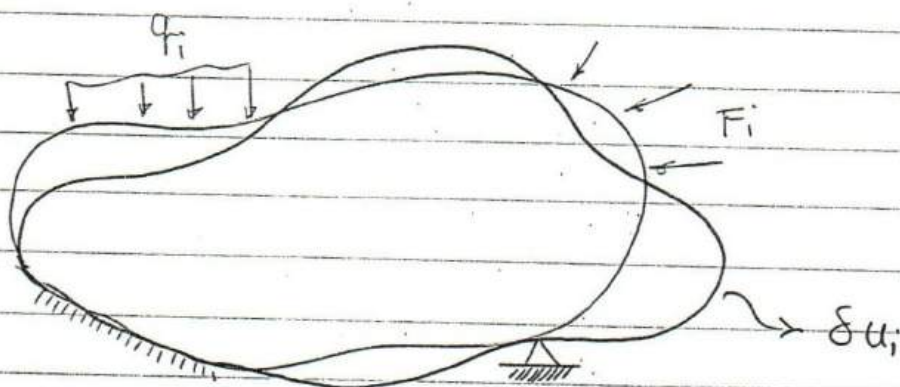
با بارها دایره حل شود.

با شکل ها غیر دایره هم حل شود.

$$\delta W_{ext} = \int_S t_i \delta u_i ds + \int_V b_i \delta u_i dv$$

$$\Rightarrow \delta W_{ext} = \delta W_{int}$$

$$\delta W_{int} = \int_V \sigma_{ij} \delta \epsilon_{ij} dv$$



$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \times \delta u_i \quad \rightarrow \quad \left(\frac{\partial \sigma_{ij}}{\partial x_j} + b_i \right) \delta u_i = 0$$

$$\int_V \rightarrow \int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + b_i \right) \delta u_i dv = 0$$

$$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i + \int_V b_i \delta u_i dv = 0$$

کار مجازی خارجی ناشی از نیروی بدنه

$$\frac{\partial}{\partial x_j} (\sigma_{ij} \delta u_i) = \left[\frac{\partial \sigma_{ij}}{\partial x_j} \cdot \delta u_i \right] + \sigma_{ij} \frac{\partial}{\partial x_j} (\delta u_i)$$

$$\rightarrow \int_V \frac{\partial}{\partial x_j} (\sigma_{ij} \delta u_i) dV = \int_V \sigma_{ij} \frac{\partial}{\partial x_j} (\delta u_i) dV + \int_V b_i \delta u_i dV = \underbrace{\delta \left(\frac{\partial u_i}{\partial x_j} \right)}_{\delta \epsilon_{ij}}$$

قسطه برین = دیورژانس

$$\int_V \nabla \cdot \mathbf{F} \cdot \mathbf{n} dV = \int_S (\sigma_{ij} \delta u_i) \cdot n_j \cdot dS = \int_S t_i \delta u_i dS$$

$\sigma_{ij} \cdot n_j = t_i$

$$\int_V \frac{\partial F}{\partial x_j} dV = \int_S F \cdot n_j dS$$

$$= \int_S t_i \delta u_i dS + \int_V b_i \delta u_i dV = \int_V \sigma_{ij} \delta \epsilon_{ij} dV$$

Betti - Maxwell Theorem

سیم پی - مالسول :

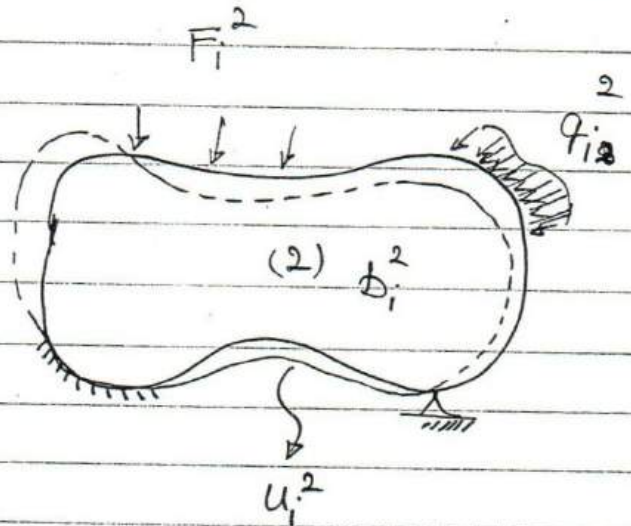
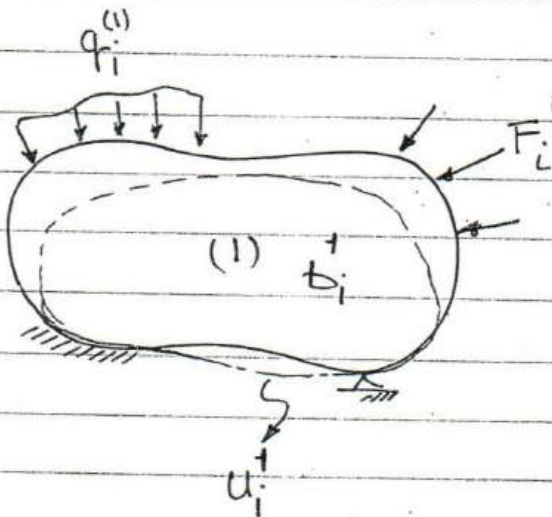
بی یک سیستم در حال تعادل تحت دو بارگذاری مختلف که منجر به دو تغییر شکل سیستم برای هر یک از دو حالت

بارگذاری می شود حاصل ضرب بارگذاری اول در جابجایی ها دوم برابر است با بارگذاری دوم در جابجایی ها

حاصل می شود

دل

$$P_{i1} \cdot u_{i2} = P_{i2} \cdot u_{i1}$$



$$\rightarrow F_i^1 \cdot u_i^2 = F_i^2 \cdot u_i^1$$

مبادرت نظر گرفتن سیستم نیرویی (1) و جابجایی 2 به عنوان میدان مجازی آن :

$$\delta W_{int}^{(1)} = \delta W_{ext}^{(1)}$$

$$\delta W_{int}^{(1)} = \int_V \sigma_{ij}^1 \epsilon_{ij}^2 dv$$

دوایر ϵ کششی نیروی حجم حول u_{i2} نیروی ستم

ف مجازی است

نیرو حول از (1) است پس σ^1 و حجم تغییر شکل جابجایی 2 ϵ^2 $P \Delta N$

$$\int_s q_i^{(1)} u_i^{(2)} ds + F_i^{(1)} u_i^{(2)}$$

$$\delta W_{ext}^1 = \int_s t_i^{(1)} u_i^{(2)} ds + \int_v b_i^{(1)} u_i^{(2)} dV$$

حالت دوم: با در نظر گرفتن سیستم نیروی 2، میدان جانبی 1 به عنوان میدان مجازی:

$$\delta W_{int}^2 = \delta W_{ext}^2$$

$$\delta W_{int}^2 = \int \sigma_{ij}^2 \cdot \epsilon_{ij}^1 dV$$

$$\delta W_{ext}^2 = \int_s t_i^2 u_i^1 ds + \int_v b_i^2 u_i^1 dV$$

$$\int q_i^2 u_i^1 ds + F_i^{(2)} u_i^{(1)}$$

$$\delta W_{int}^1 = \int \sigma_{ij}^1 \cdot \epsilon_{ij}^2 dV$$

$$\sigma_x = E \cdot \epsilon_x$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underline{D} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \rightarrow \{\sigma\} = [D] \{\epsilon\}$$

$$\delta W_{int}^1 = \int_V \underline{\sigma}_{ij}^1 \underline{\epsilon}_{ij}^2 dV = \int_V \underline{\epsilon}_{ij}^{(2)T} \underline{\sigma}_{ij}^1 dV$$

$$= \int_V \underline{\epsilon}_{ij}^{(2)T} \underline{D}^x \underline{\epsilon}_{ij}^1 dV$$

ماتریس خواص ماده است و برای همه یکی است $\Rightarrow \underline{D}^1 = \underline{D}^2 = \underline{D}$

$$\delta W_{int}^2 = \int_V \underline{\sigma}_{ij}^2 \underline{\epsilon}_{ij}^1 dV = \int_V \underline{\epsilon}_{ij}^{(1)T} \underline{\sigma}_{ij}^2 dV$$

$$= \int_V \left[\underline{\epsilon}_{ij}^{(1)T} \underline{D}^x \underline{\epsilon}_{ij}^2 \right] dV$$

چون کار اسکلر است پس این دو تانسور نیز هم‌اندازه و هم‌جهت خواهند بود.

$$\underline{\epsilon}_{ij}^{(2)T} \underline{D}^T \underline{\epsilon}_{ij}^{(1)} \rightarrow \underline{D}$$

$$\rightarrow \delta W_{int}^1 = \delta W_{int}^2 \rightarrow \delta W_{ext}^1 = \delta W_{ext}^2$$

$$\frac{\delta \sigma_{ij}}{\delta x_j} + b_i = 0$$

$$\delta W_{int} = \delta W_{ext}$$

مقدار جانبی مجازی δu_i

جانبی دگرخواه ← اختیاری

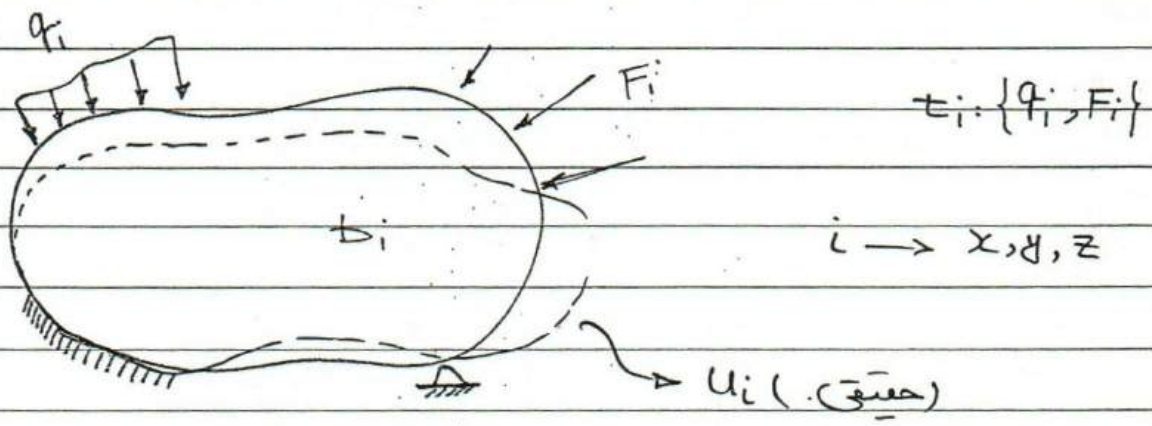
u_i ← جانبی حقیقی؟

ما از میان جانبی های مجازی دنبال جانبی درست حقیقی است.

مقدار انرژی پتانسیل مینیمم:

مطابق این مقدار از بین تمام جانبی های دگرخواه که می توان برای یک سیستم در حال تعادل در نظر گرفت

تنها جانبی حقیقی انرژی پتانسیل کل سیستم را مینیمم می کند.



از میان تمام جانبی های مجازی فقط یکی است که جانبی درست سیستم است

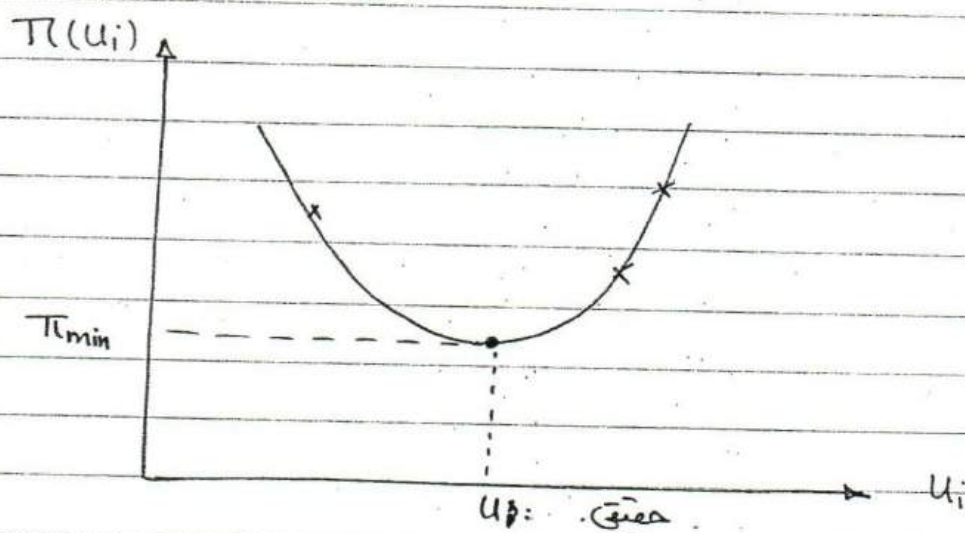
$$\pi(u_i) \Rightarrow \text{انرژی پتانسیل کل سیستم}$$

از میان تمام حالت‌های مجازی فقط یکی حقیقی است. و آن آنی است که انرژی پتانسیل کل سیستم را

$$\rightarrow \delta \pi_i = 0$$

مینمی

$$\frac{\delta \pi_i}{\delta u_i} \cdot \delta u_i = 0$$



$$\pi(u_i) = U(u_i) + V(u_i) = \text{انرژی پتانسیل کل سیستم}$$

$$\pi(u_i) = U(u_i) + V(u_i)$$

معادل است با کار خارجی با علامت منفی

$$V = -W_{\text{ext}}$$

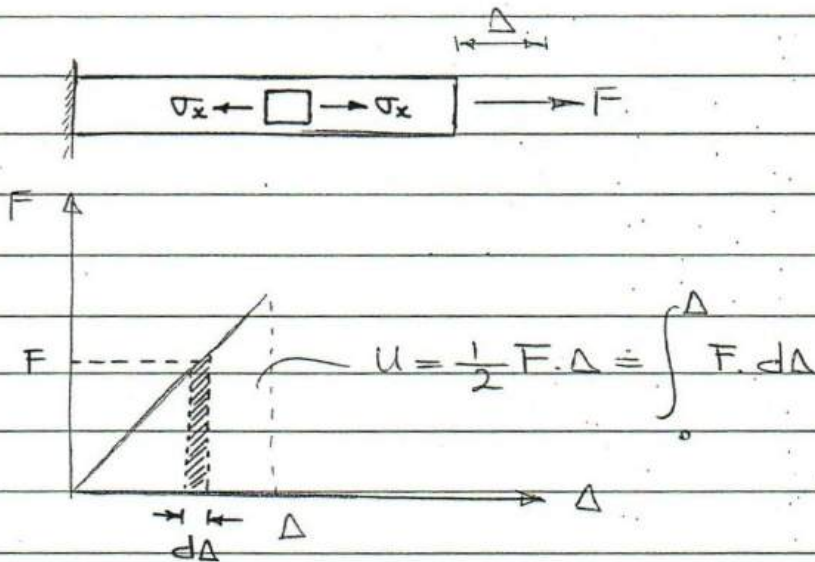
انرژی پتانسیل بیرونی (کرنه خارجی) = -

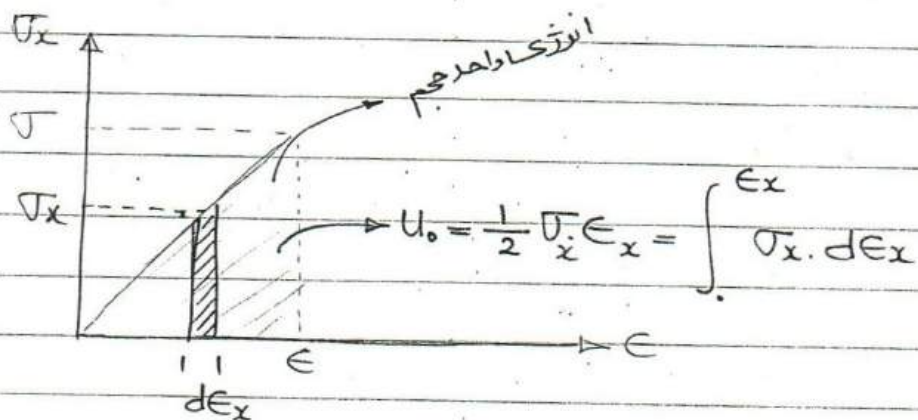
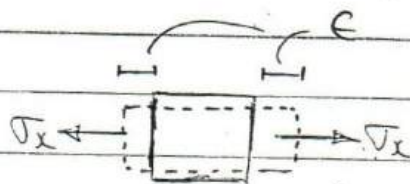
$$V = -W_{ext}$$

$$= - \int_S t_i u_i ds - \int_V b_i u_i dV$$

$$\Rightarrow V = -W_{ext} = - \int_S t_i u_i ds - \int_V b_i u_i dV$$

انرژی داخلی سیستم = U





$$u_0 = \frac{U}{V} = \frac{1}{2} \frac{(\sigma_x \epsilon_x) (F \Delta)}{A L}$$

$$u_0 = \frac{1}{2} \sigma_x \epsilon_x + \frac{1}{2} \sigma_y \epsilon_y$$

$$\text{Potential Energy} = U = \int_V u_0 dV = \int_V \frac{1}{2} \sigma_{ij} \epsilon_{ij} dV$$

$$u_0 = \frac{1}{2} \sigma_{ij} \epsilon_{ij}$$

$$\Pi = U + V = \int_V u_{(u)} dV - \int_S t_i u_i ds - \int_V b_i u_i dV$$

$$\delta \Pi = \int_V \left[\frac{\delta u_0}{\delta \epsilon_{ij}} \delta \epsilon_{ij} + \frac{\delta \epsilon_{ij}}{\delta u_i} \delta u_i \right] dV - \int_S t_i \delta u_i ds - \int_V b_i \delta u_i dV = 0$$

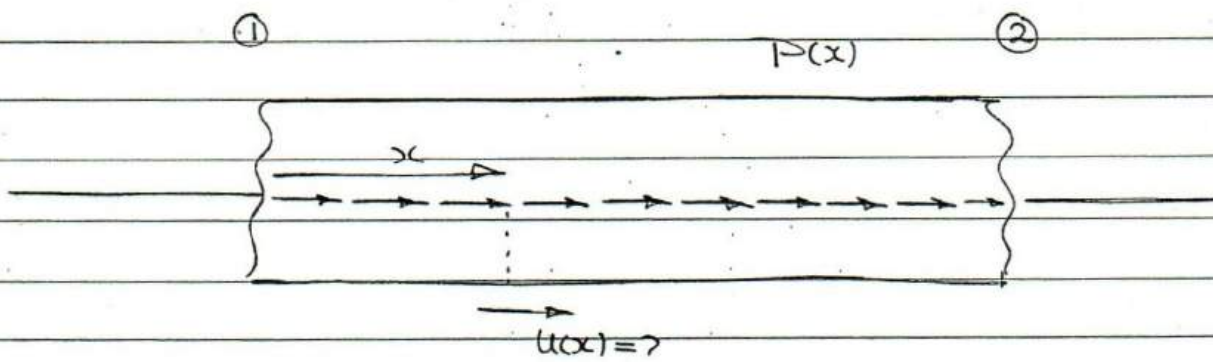
$\underbrace{\quad}_{\delta W_{int}} \quad \underbrace{\quad}_{\delta W_{ext}}$

$$\delta W_{int} - \delta W_{ext} = 0$$

مثال: برای عضو محوری شکل زیر که تحت بار با شدت $P(x)$ قرار گرفته مطابقت بگیرد. معادله تعادل را بنویسید.

$$\sum F_x = 0 \Rightarrow \text{واقعاً است} \quad \text{(معادله تعادل از قیاس انرژی به دست می آید)}$$

$$\int_0^L P(x) dx = 0$$



$$\pi_{(u(x))} = U + V \rightsquigarrow \delta \pi = 0 \Rightarrow u(x)$$

$$V \Rightarrow \text{انرژی پتانسیل نیروی خارجی} \Rightarrow V_{(u(x))} = \int_L P(x) \cdot u(x) dx$$

$$U \Rightarrow \text{انرژی داخلی سیستم} \Rightarrow U = \int_V \frac{1}{2} \sigma_{ij} \epsilon_{ij} dv$$

$$\Rightarrow U = \int_V \frac{1}{2} \sigma_x \cdot \epsilon_x dv = \int_L \frac{1}{2} E \underbrace{\epsilon_x^2}_{\left(\frac{du}{dx}\right)^2} A dx$$

$$\rightarrow \epsilon_x = \frac{du}{dx}$$

$$U = \int_L \frac{1}{2} AE \cdot \left(\frac{du(x)}{dx}\right)^2 dx$$

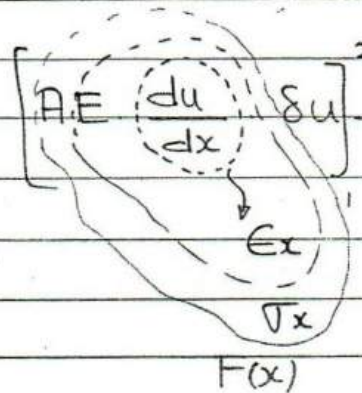
$$\pi_{(u(x))} = U + V = \int_L \frac{1}{2} AE \cdot \left(\frac{du(x)}{dx}\right)^2 dx - \int_L P(x) u(x) dx$$

$$\delta \pi = 0 \Rightarrow \int_L \frac{1}{2} AE \cdot 2 \left(\frac{du}{dx}\right) \underbrace{\delta \left(\frac{du}{dx}\right)}_{\frac{d}{dx}(\delta u)} dx$$

$$- \int_L P \delta u \cdot dx = 0$$

$$\int \underbrace{AE \cdot \frac{du}{dx}}_u \underbrace{\frac{d}{dx}(\delta u)}_{dv} dx - \int p(x) \delta u \cdot dx = 0$$

استدلال لایبری خرد و بجز

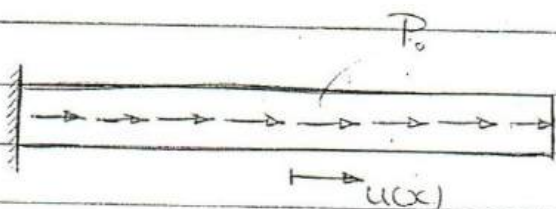


$$\left[AE \frac{du}{dx} \delta u \right]_1^2 - \int_1^2 AE \frac{d^2 u}{dx^2} \delta u \cdot dx - \int p(x) \delta u \cdot dx = 0$$

$$\left[\cancel{F_2} \cdot \delta u_2 - \cancel{F_1} \cdot \delta u_1 \right] - \int_L \left[AE \frac{d^2 u}{dx^2} + p(x) \right] \delta u(x) \cdot dx = 0$$

$$\left[AE \frac{d^2 u}{dx^2} + p(x) \right] \delta u(x) = 0 \xrightarrow{\delta u(x) \neq 0} AE \frac{d^2 u}{dx^2} + p(x) = 0$$

$$AE \frac{d^2 u(x)}{dx^2} + p(x) = 0 \quad \text{معادله دینامیک استیل در عضو عضو}$$



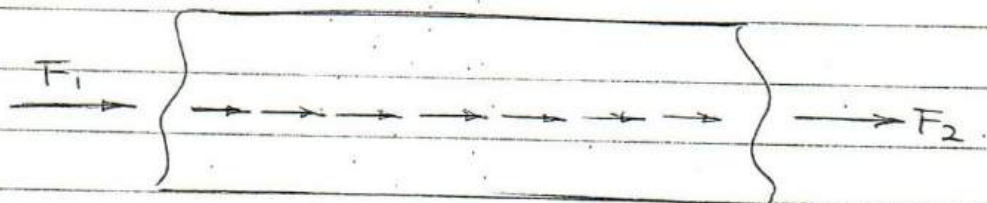
$$\frac{d^2 u}{dx^2} = -\frac{P_0}{AE} \rightarrow \frac{du}{dx} = -\frac{P_0}{AE} x + C_1$$

$$\Rightarrow u(x) = -\frac{P_0}{2AE} x^2 + C_1 x + C_2$$

$$\text{B.C.1} \Rightarrow u(x) \Big|_{x=0} = 0 \rightarrow C_2 = 0$$

$$\text{B.C.2} \Rightarrow \sigma_x = E \cdot \epsilon_x = E \cdot \frac{du}{dx} \Big|_{x=L} = 0 \rightarrow C_1 = \frac{P_0 L}{AE}$$

شیرین



$$= \int_L \int_A \frac{1}{2} E \left(\frac{du}{dx} \right)^2 dA dx = \int_L \int_A \frac{1}{2} E \left(-y \frac{d^2 v}{dx^2} \right)^2 dA dx$$

$$U = \int_L \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx \quad \text{انرژی داخلی عضو خمی}$$

$$\Pi = \int_L \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx - \int q(x) v(x) dx$$

$$\delta \Pi = 0 \rightarrow \text{تغییرات نسبت به } v(x)$$

$$\int_L \frac{1}{2} EI \cancel{2} \left(\frac{d^2 v}{dx^2} \right) \delta \left(\frac{d^2 v}{dx^2} \right) dx - \int_L q(x) \delta v(x) dx =$$

$$\int_L \underbrace{EI \frac{d^2 v}{dx^2}}_{\delta u} \cdot \underbrace{\frac{d^2}{dx^2} (\delta v)}_{\delta v} dx - \int q(x) \delta v(x) dx =$$

انتگرال لایبزنیز جزئی

$$\left[\underbrace{EI \frac{d^2 v}{dx^2}}_{-M(x)} \cdot \underbrace{\frac{d}{dx} (\delta v)}_{\delta \theta(x)} \right]_1 - \int_L \underbrace{EI \frac{d^3 v}{dx^3}}_u \cdot \underbrace{\frac{d}{dx} (\delta v)}_{\delta v} dx - \int q(x) \delta v(x) dx =$$

$$\text{PAN} \quad \frac{d}{dx} \delta v \cdot \delta \left(\frac{dv}{dx} \right) \cdot \delta \theta$$

$$\begin{aligned}
 & \cancel{M_1} (M_1 \delta \theta_1 - M_2 \delta \theta_2) - \left[\left(EI \frac{d^3 V}{dx^3} \right) \delta V(x) \right]^2 + \int EI \frac{d^4 V}{dx^4} \delta V dx \\
 & - \int q(x) \delta V \cdot dx = 0
 \end{aligned}$$

$$(M_1 \delta \theta_1 - M_2 \delta \theta_2) + \left(\frac{1}{2} \delta V_2 - \frac{1}{2} \delta V_1 \right) + \int \left[EI \frac{d^4 V}{dx^4} - q(x) \right] \delta V(x) dx = 0$$

$$\int \left[EI \frac{d^4 V}{dx^4} - q(x) \right] \delta V(x) dx = 0 \quad \rightarrow \quad \delta V(x) = 0$$

$$\Rightarrow EI \frac{d^4 V(x)}{dx^4} - q(x) = 0$$

معادله دینامیک عضو حقیقی

حالت دوم ← معادله حقیقی

$$u = \int_V \frac{1}{2} \sigma_x \epsilon_x dV + \int_V \frac{1}{2} \sigma_x \epsilon_x dx$$

$\downarrow$ $\quad \quad \quad \downarrow$
 معادله $\quad \quad \quad$ حقیقی
 $AE \frac{d^2 u}{dx^2} \quad \quad \quad EI \frac{d^4 u}{dx^4}$

اثر تمام خمشی و برشی:

$$U = \int_V \frac{1}{2} \sigma_{ij} \epsilon_{ij} dV = \int_V \frac{1}{2} \sigma_x \epsilon_x dV + \underbrace{\int_V \frac{1}{2} \tau_{xy} \gamma_{xy} dV}_{\int_V \frac{1}{2} \cdot \frac{\tau^2}{G} dV}$$

$$\tau = \frac{V \cdot Q}{I \cdot t} \quad \rightarrow \quad V(x) = -EI \cdot \frac{d^3 u(x)}{dx^3}$$

$$\int_V \frac{1}{2} \cdot \frac{V^2 Q^2}{I^2 \cdot t^2 G} dV$$

$$\iint_{L \cdot A} \frac{1}{2} \cdot \frac{E^2 \cdot I^2 \cdot Q^2}{I^2 \cdot t^2 \cdot G} \cdot \left(\frac{d^3 u}{dx^3} \right)^2 dA dx$$

قانون هooke $\Rightarrow U = \int_L \frac{1}{2} AE \left(\frac{du}{dx} \right)^2 dx$

قانون هooke $\Rightarrow U = \int_L \frac{1}{2} EI \left(\frac{d^2 u}{dx^2} \right)^2 dx$

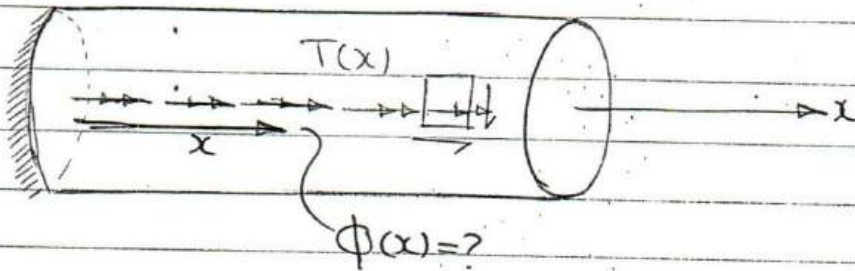
قانون هooke $U = \int_L \frac{1}{2} G \cdot A' \left(\frac{d^3 u}{dx^2} \right)^2 dx$

PAN

$$A' = \int_A \frac{E^2 Q^2}{t^2 \cdot G^2} \cdot dA$$

48.

در صورتی که اثر پیچش نیز در نظر گرفته شود، برای سادگی عضو دایره‌ای



$$V = \int T(x) \cdot \phi(x) \cdot dx$$

$$U = \int \frac{1}{2} \tau_{ij} \epsilon_{ij} dV = \int \frac{1}{2} \tau \gamma dV = \int \int \frac{1}{2} \frac{\tau^2}{G} dA \cdot dx$$

$\tau_{xy} \cdot \gamma_{xy}$ τ $\frac{\tau}{G}$

$$\tau_{\max} = \frac{T \cdot R}{J}, \quad d\phi \cdot R = \gamma_{\max} \cdot dx$$

$$\frac{\tau_{\max}}{G} \rightarrow \frac{T \cdot R}{J}$$

$$d\phi \cdot R = \frac{T \cdot R}{G \cdot J} dx \Rightarrow \frac{d\phi}{dx} = \frac{T}{G \cdot J} \Rightarrow T = G \cdot J \frac{d\phi}{dx}$$

$$U = \int \int \frac{1}{2} \frac{\tau^2 \cdot r^2}{G \cdot J^2} dA \cdot dx = \int \int \frac{1}{2} \frac{G \cdot J^2 \cdot r^2}{G \cdot J^2} \left(\frac{d\phi}{dx} \right)^2 dA \cdot dx$$

$$U = \int_L \frac{1}{2} \cdot G \cdot J \left(\frac{d\varphi(x)}{dx} \right)^2 \cdot dx$$

انرژی داخلی عضو پیچشی

$$\Pi = \int_L \frac{1}{2} \cdot G \cdot J \left(\frac{d\varphi}{dx} \right)^2 \cdot dx - \int T(x) \cdot \varphi(x) \cdot dx$$

$$\delta \Pi = \dots \rightarrow \varphi(x) = \dots$$

$$GJ \frac{d^2 \varphi}{dx^2} + T(x) = \dots \rightarrow \text{معادله دینامیک عضو پیچشی}$$

بامقطع دایروی.

$\epsilon_{ij} =$ ← یا کامل با سفتی محوری بی نهایت ←

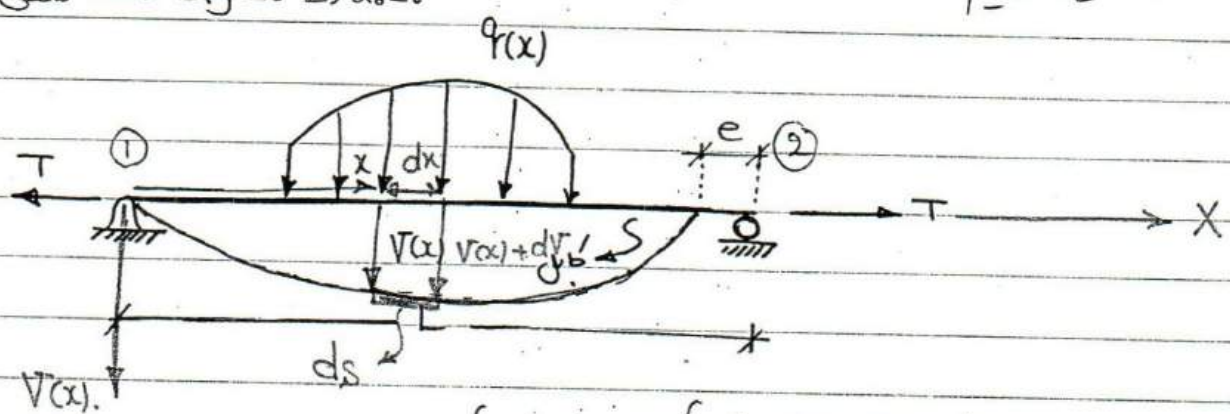
ل. کامل مطابق شکل (بدون سفتی) در نظر گرفته که تحت بار عمودی $q(x)$ قرار دارد برای ایجاد جابجایی در سیستم

$\epsilon = 0 \Rightarrow \sigma = 0$

تغییری ناشی از آن به توانیهای آن واردی شود. مطلوب است تعیین تغییر شکل کامل با استفاده

$u_0 = 0 \Rightarrow v = 0 \Rightarrow \epsilon = 0$ بدون سفتی

تغییر انرژی پتانسیل سیستم $\epsilon_{ij} = 0 \Rightarrow u_0 = 0$ سفتی بی نهایت



$\pi = u + v \rightarrow u = \int_V u \cdot dv = \int \frac{1}{2} \sigma_{ij} \epsilon_{ij} dv$

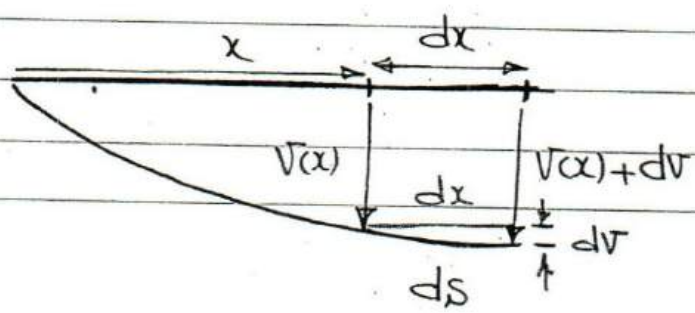
$\Rightarrow \delta \pi = 0$

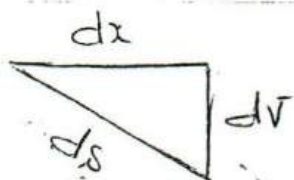
$v = - \int t_i u_i ds - \int b_i u_i dv$

$$\begin{cases} u = 0 \\ v_{(V(x))} = \int q(x) \cdot V(x) \cdot dx - T(-e) \end{cases} \rightarrow e_{(V(x))}$$

$\delta \pi = 0 \rightarrow V(x) = \dots$

$e = \int_l (ds - dx)$





$$ds^2 = dv^2 + dx^2 \Rightarrow ds^2 = \left[\left(\frac{dv}{dx} \right)^2 + 1 \right] dx^2$$

$$ds = \sqrt{\left(\frac{dv}{dx} \right)^2 + 1} \cdot dx$$

$$e = \int_l ds - dx = \int_l \left[\sqrt{\left(\frac{dv}{dx} \right)^2 + 1} - 1 \right] dx$$

$$\text{سب-تیلور} \Rightarrow \left[1 + \frac{1}{2} \frac{d^2 v}{dx^2} + \dots - 1 \right] dx$$

1/5
سب-سری تیلور، یاد دہن (تفصیل) بنائیں۔

$$e = \int_l \frac{1}{2} \cdot \frac{d^2 v}{dx^2} \cdot dx$$

$$\pi = u + v = - \int_l q(x) \cdot v(x) \cdot dx + \int \frac{T}{2} \cdot \frac{d^2 v(x)}{dx^2} \cdot dx$$

$$\delta \pi = - \int_l q(x) \cdot \delta v(x) \cdot dx + \int T \cdot \underbrace{\frac{dv}{dx}}_u \cdot \underbrace{\delta \left(\frac{dv}{dx} \right)}_{dv} \cdot dx$$

$\nearrow \frac{d}{dx} \delta v$

$$-\int_L q(x) \cdot \delta V \cdot dx + T \cdot \frac{dV}{dx} \cdot \delta V(x) \Big|_1^2 - \int_L T \cdot \frac{d^2 V}{dx^2} \cdot \delta V \cdot dx = 0$$

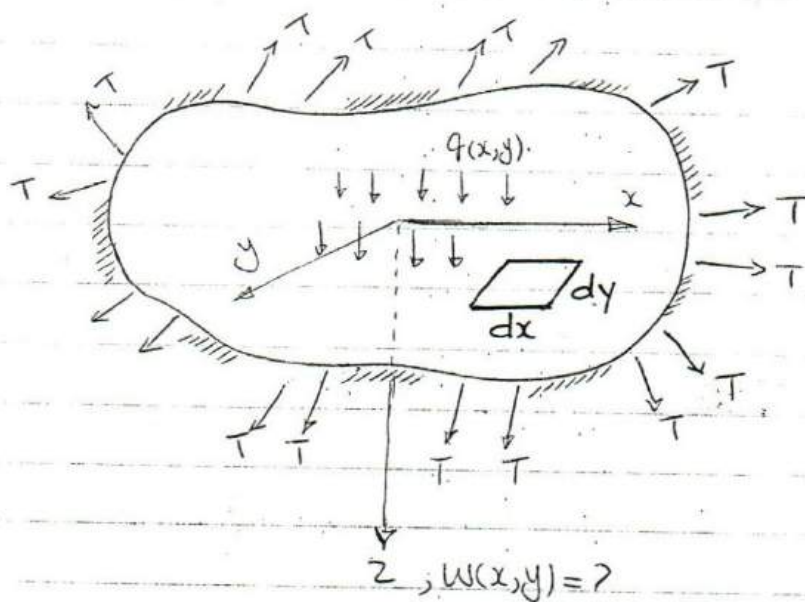
$$-\int_L \left[T \cdot \frac{d^2 V}{dx^2} + q(x) \right] (\delta V) dx + T \cdot \left(\frac{dV}{dx} \right)_2 \cdot \delta V_2 - T \cdot \left(\frac{dV}{dx} \right)_1 \cdot \delta V_1 = 0$$

$$\delta V \neq 0 \rightarrow T \cdot \frac{d^2 V}{dx^2} + q(x) = 0 \Rightarrow \text{معادله دیفرانسیل تغییر شکل کابل}$$

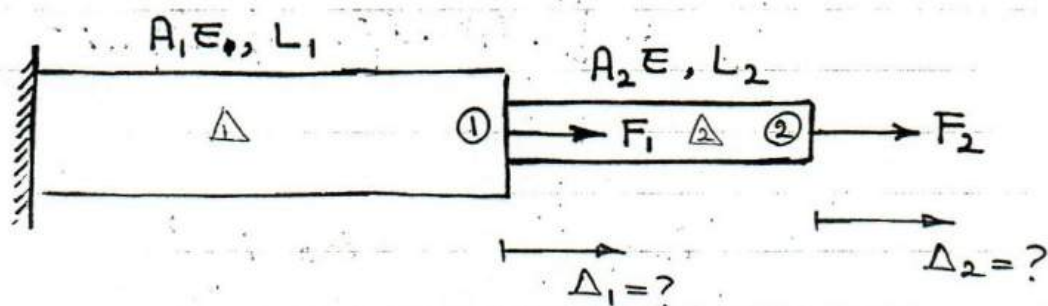
نکته:

در نظر گرفتن غشایی بدون سختی مطابق شکل که تحت بار عرضی $q(x, y)$ قرار گرفته و جهت تعادل شاکش T از هر طرف به غشادر تراز افقی وارد می شود. مطلوب است تعیین تغییر شکل

غشا با استفاده از قضیه انرژی پتانسیل مینیم



مثال: برای عضو مخوری شکل زیر که از دو قسمت تشکیل شده است دو بارگذاری متمرکز F_1 و F_2 قرار گرفته
 مطلوب است تعیین جابه جایی نقاط 1 و 2 با استفاده از قضیه انرژی پتانسیل مینیمم



معمولات Δ_1 و Δ_2 هستند

$$\Pi = U + V \rightarrow \Pi(\Delta_1, \Delta_2) = U(\Delta_1, \Delta_2) + V(\Delta_1, \Delta_2)$$

$$\delta \Pi = 0 \Rightarrow \frac{\partial \Pi}{\partial \Delta_1} \delta \Delta_1 + \frac{\partial \Pi}{\partial \Delta_2} \delta \Delta_2 = 0$$

$$\rightarrow \delta \Delta_1 \neq 0 \Rightarrow \delta \Delta_2 \neq 0 \Rightarrow \begin{cases} \frac{\partial \Pi}{\partial \Delta_1} = 0 \\ \frac{\partial \Pi}{\partial \Delta_2} = 0 \end{cases} \Rightarrow \Delta_1 \checkmark, \Delta_2 \checkmark$$

$$\rightarrow K_1 = \frac{A_1 E}{L_1}, \quad K_2 = \frac{A_2 E}{L_2}$$

$$U = U_1 + U_2 = U_1 + U_2 = \frac{1}{2} K_1 \Delta_1^2 + \frac{1}{2} K_2 (\Delta_2 - \Delta_1)^2$$

$$U = \int \frac{1}{2} \sigma_x \epsilon_x dV = \int \frac{1}{2} \cdot E \cdot \epsilon_x^2 dV = \int \frac{1}{2}$$

$$U = \iint \frac{1}{2} \sigma_x \cdot \epsilon_x \cdot dA \cdot dx = \int \frac{1}{2} \cdot A \cdot E \cdot \epsilon_x^2 dx$$

$$= \int \frac{1}{2} \cdot A \cdot E \cdot \frac{\Delta^2}{L^2} dx$$

$$= \frac{1}{2} \cdot A E \cdot \frac{\Delta^2}{L} = \frac{1}{2} \cdot \frac{A E}{L} \cdot \Delta^2$$

$$\rightarrow U = U_{(\Delta_1, \Delta_2)}^{\Delta_1} + U_{(\Delta_1, \Delta_2)}^{\Delta_2} = \frac{1}{2} K_1 \cdot \Delta_1^2 + \frac{1}{2} K_2 (\Delta_2 - \Delta_1)^2$$

$$\rightarrow V = -F_1 \Delta_1 - F_2 \Delta_2$$

$$\Pi = U + V = \frac{1}{2} K_1 \Delta_1^2 + \frac{1}{2} K_2 (\Delta_2 - \Delta_1)^2 - F_1 \Delta_1 - F_2 \Delta_2$$

$$\Pi = \frac{1}{2} K_1 \Delta_1^2 + \frac{1}{2} K_2 \Delta_2^2 + \frac{1}{2} K_2 \Delta_1^2 - K_2 \Delta_1 \Delta_2 - F_1 \Delta_1 - F_2 \Delta_2$$

$$\frac{\partial \Pi}{\partial \Delta_1} = 0 \rightarrow K_1 \Delta_1 + K_2 \Delta_1 - K_2 \Delta_2 - F_1 = 0$$

$$\frac{\partial \Pi}{\partial \Delta_2} = 0 \rightarrow K_2 \Delta_2 - K_2 \Delta_1 - F_2 = 0$$

displacement vector.

$$\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness matrix

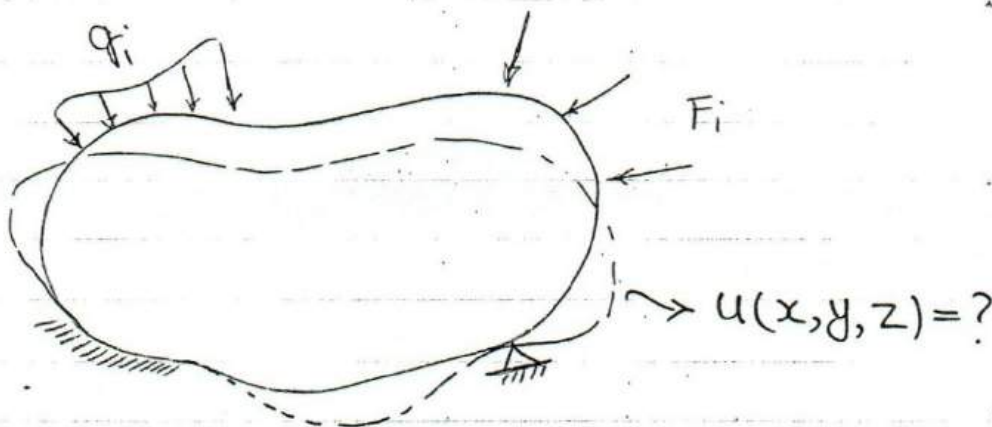
Force vector

Rayleigh - Ritz

روش ریلی - ریتز

میدان جابه جایی

مطابق این روش هر تابع دلخواه را می توان بر اساس مجموعه ای از یک سری توابع معین (از پیش تعیین شده) و ضرایب مجهول بیان نمود.



مجهول

$$u(x, y, z) = \sum_{i=1}^{\infty} f_i(x, y, z) \cdot a_i$$

$$\approx \sum_{i=1}^n f_i(x, y, z) \cdot a_i$$

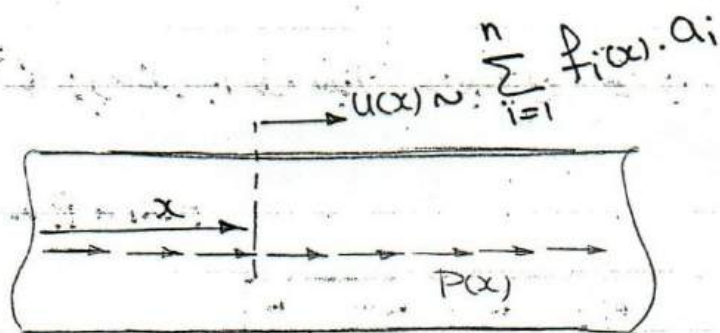
$$\pi: \begin{cases} u_{(a_i)} = \int_V u \cdot dV = \int_V \frac{1}{2} \nabla_{ij} \epsilon_{ij}^{(a_i)} dV \\ V_{(a_i)} = - \int_S t_i u_{i(a_i)} ds - \int_V b_i u_{i(a_i)} dV \end{cases}$$

$$\frac{\delta \pi}{\delta a_1} \delta a_1 + \frac{\delta \pi}{\delta a_2} \delta a_2 + \dots + \frac{\delta \pi}{\delta a_i} \delta a_i = 0$$

$$\begin{cases} \frac{\delta \pi}{\delta a_1} = 0 \\ \frac{\delta \pi}{\delta a_2} = 0 \\ \vdots \\ \frac{\delta \pi}{\delta a_i} = 0 \end{cases}$$

$$a_1, a_2, \dots, a_i$$

$$\pi: \begin{cases} u_{(a_i)} = \int_V u \cdot dV = \int_V \frac{1}{2} \nabla_{ij} \epsilon_{ij}^{(a_i)} dV \\ V_{(a_i)} = - \int_S t_i u_{i(a_i)} ds - \int_V b_i u_{i(a_i)} dV \end{cases}$$



$$U = \int \frac{1}{2} AE \cdot \left(\frac{du(x)}{dx} \right)^2 \cdot dx$$

$$V = - \int P(x) \cdot u(x) \cdot dx$$

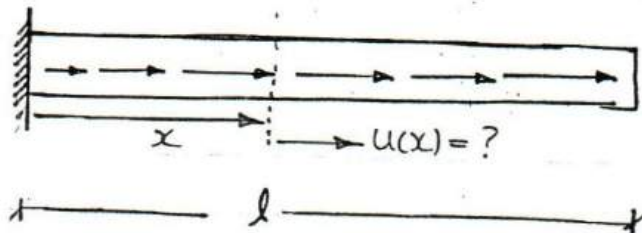
$$u(x) \sim \sum_{i=1}^n f_i(x) \cdot a_i$$

$$\Pi(u_i) = \int_l \frac{1}{2} AE \left[\sum_{i=1}^n \frac{df_i}{dx} \cdot a_i \right]^2 dx - \int_l P(x) \left[\sum f_i(x) a_i \right] dx$$

$$\rightarrow \delta \Pi = 0 \rightarrow a_i = \sqrt{\quad}$$

برای عضو محوری شکل زیر مطلوب است میدان جابجایی افقی با استفاده از روش ریلی - رینز

$$P(x) = P_0 x$$



$$u(x) \approx \sum_{i=1}^n f_i(x) \cdot a_i$$

$$\pi_{(a_i)} = U + V = \int_L \frac{1}{2} AE \left(\frac{du(x)}{dx} \right)^2 dx - \int P(x) \cdot u(x) \cdot dx$$

$$\pi = \int_L \frac{1}{2} AE \cdot \left[\sum_{i=1}^n \frac{df_i(x)}{dx} \cdot a_i \right]^2 dx - \int P(x) \left[\sum_{i=1}^n f_i(x) \cdot a_i \right] dx$$

چون عدد تابع است، عدد تابع تغییر ندارد؟؟ «از نظر ریاضی»

$$\delta \pi = \frac{\partial \pi}{\partial a_i} \delta a_i = 0 \Rightarrow \underbrace{\delta a_i \neq 0}_{\text{جدا}} \Rightarrow \frac{\partial \pi}{\partial a_i} = 0 \Rightarrow a_i = ?$$

Try 1) $\leadsto u_1(x) = a_0 + a_1 x$

$$x = 0 \Rightarrow u_1(0) = 0 \Rightarrow a_0 = 0 \leadsto u_1(x) = a_1 x$$

$$U_1 = \int_L \frac{1}{2} AE \cdot (a_1)^2 dx = \frac{1}{2} EA \cdot a_1^2 l$$

$$V_1 = - \int (P_0 x) (a_1 x) dx = - \frac{P_0 a_1 l^3}{3}$$

$$\pi = U_1 + V_1 = \frac{1}{2} AE \cdot a_1^2 l - \frac{P \cdot a_1 L^3}{3}$$

$$\delta \pi = \frac{d\pi}{da_1} \cdot \delta a_1 \stackrel{\neq 0}{=} 0 \Rightarrow AE a_1 l - \frac{P L^3}{3} \Rightarrow a_1 = \frac{P L^2}{3 EA}$$

$$\rightarrow u_1(x) = \frac{P L^2}{3 AE} x$$

$$\text{Try 2: } u_2(x) = a_0 + a_1 x + a_2 x^2$$

$$x_0 \Rightarrow u_2 = 0 \Rightarrow a_0 = 0$$

$$U_2 = \int_0^L \frac{1}{2} AE (a_1 + 2a_2 x)^2 dx = \frac{1}{2} AE (a_1^2 l + \frac{4}{3} a_2^2 L^3 + 2a_1 a_2 L^2)$$

$$a_1^2 + 4a_2 a_1 x + 4a_2^2 x^2$$

$$V_2 = - \int_0^L P x \cdot (a_1 x + a_2 x^2) dx = -P a_1 \cdot \frac{L^3}{3} - P a_2 \cdot \frac{L^4}{4}$$

$$\pi_{(a_1, a_2)} = U_2 + V_2 = \frac{1}{2} AE (a_1^2 L + \frac{4}{3} a_2^2 L^3 + 2a_1 a_2 L^2) - \frac{1}{3} P a_1 L^3 - \frac{1}{4} P a_2 L^4$$

$$\delta \pi = 0 \Rightarrow \frac{\delta \pi}{\delta a_1} (\delta a_1) + \frac{\delta \pi}{\delta a_2} (\delta a_2) = 0 \Rightarrow \frac{\delta \pi}{\delta a_1} = 0$$

$$\frac{\delta \pi}{\delta a_2} = 0$$

$$\frac{1}{2} AE (2a_1 l + 2a_2 l^2) - \frac{P \cdot L^3}{3} = 0 \quad \leftarrow \frac{\delta \pi}{\delta a_1} = 0$$

$$\frac{1}{2} AE \left(\frac{8}{3} a_2 L^3 + 2a_1 l^2 \right) - \frac{P \cdot L^4}{4} = 0 \quad \leftarrow \frac{\delta \pi}{\delta a_2} = 0$$

$$a_1 = \frac{7 P \cdot L^2}{12 AE} \quad a_2 = \frac{-P \cdot L}{4 AE}$$

$$\rightarrow u_2(x) = \frac{7 P \cdot L^2}{12 AE} x - \frac{P \cdot L}{4 AE} x^2$$

(حل دقيق): $AE \cdot \frac{d^2 u}{dx^2} + P(x) = 0$ $\nearrow P \cdot x$

$$\frac{d^2 u}{dx^2} = -\frac{P \cdot x}{AE} \quad \rightsquigarrow \frac{du}{dx} = -\frac{P}{2EA} x^2 + C_1$$

$$\rightsquigarrow u(x) = -\frac{P}{6EA} x^3 + C_1 x + C_2$$

B.C 1) $\rightarrow u(x) \Big|_{x=0} = 0 \rightarrow C_2 = 0$

BC 2) $\rightarrow T = E \cdot \frac{du}{dx} \Big|_{x=l} = 0 \rightarrow C_1 = \frac{P \cdot L^2}{2AE}$

$$\rightarrow u_{\text{exact}}(x) = -\frac{P}{6AE} x^3 + \frac{P \cdot L^2}{2AE} x$$

25.

Try 3): $U_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$$\frac{\delta \pi}{\delta a_1} = 0$$

$$a_1 = \frac{P \cdot L^2}{2EA}$$

$$\frac{\delta \pi}{\delta a_2} = 0$$

$$a_2 = 0$$

$$\frac{\delta \pi}{\delta a_3} = 0$$

$$a_3 = \frac{-P}{6EA}$$

$$U_3(x) = \frac{-P}{6AE}x^3 + \frac{P \cdot L^2}{2EA}x$$

حل شود

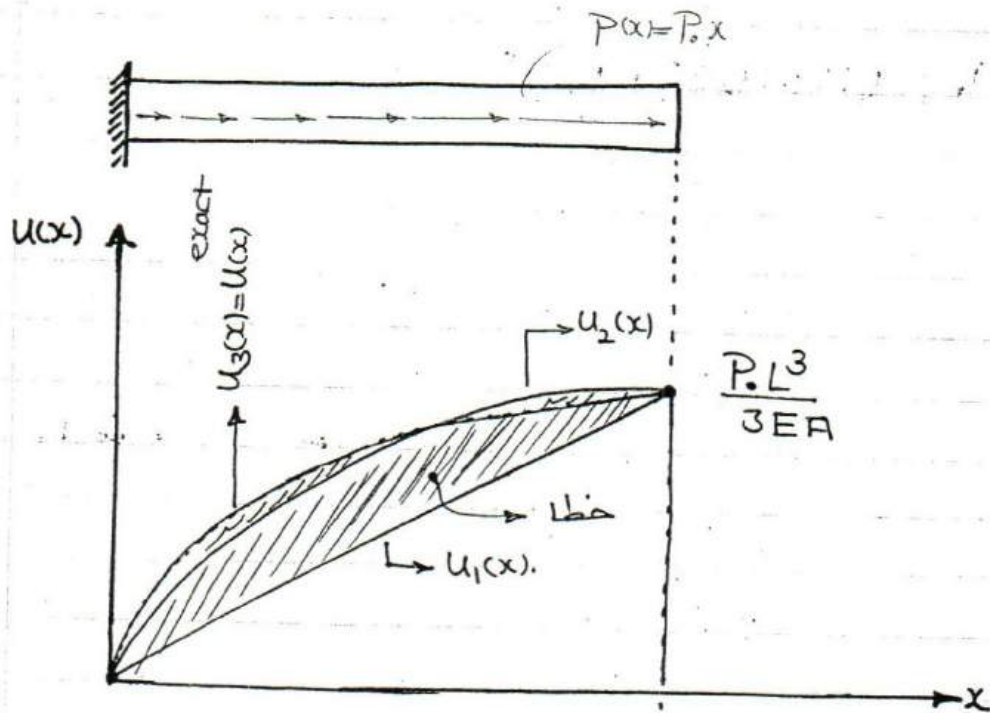
Try 4) $U_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$$\frac{\delta \pi}{\delta a_1} = 0$$

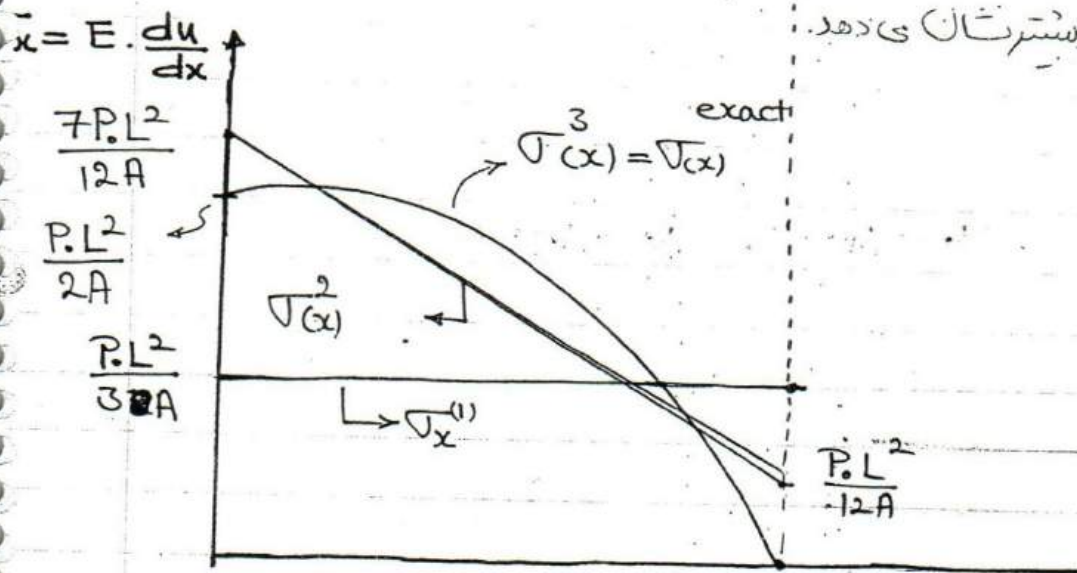
$$\frac{\delta \pi}{\delta a_4} = 0$$

$$a_4 = 0$$



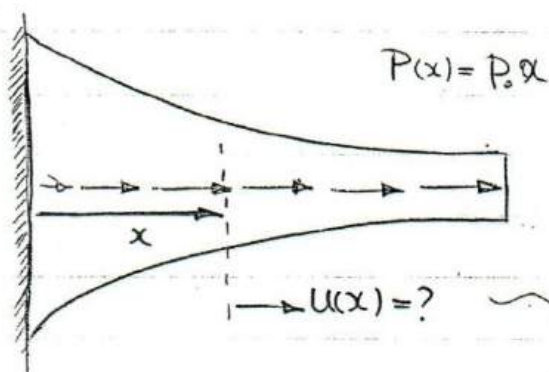


ی مقایسه خطا هیچگاه تغییر مکان را مبنای قرار ندهید. تنش را ملاک قرار دهید چون $\sigma \propto E$ و ϵ متنوع
بر مکان است و متنوع خطا را بهتر و بیشتر نشان می دهد.



$$\sigma_x^2(x) = \frac{7P_0 L^2}{12A} - \frac{P_0}{2}x$$

$$\sigma_x^3(x) = \frac{-P_0}{2A}x^2 + \frac{P_0}{2}x$$



Try 1:

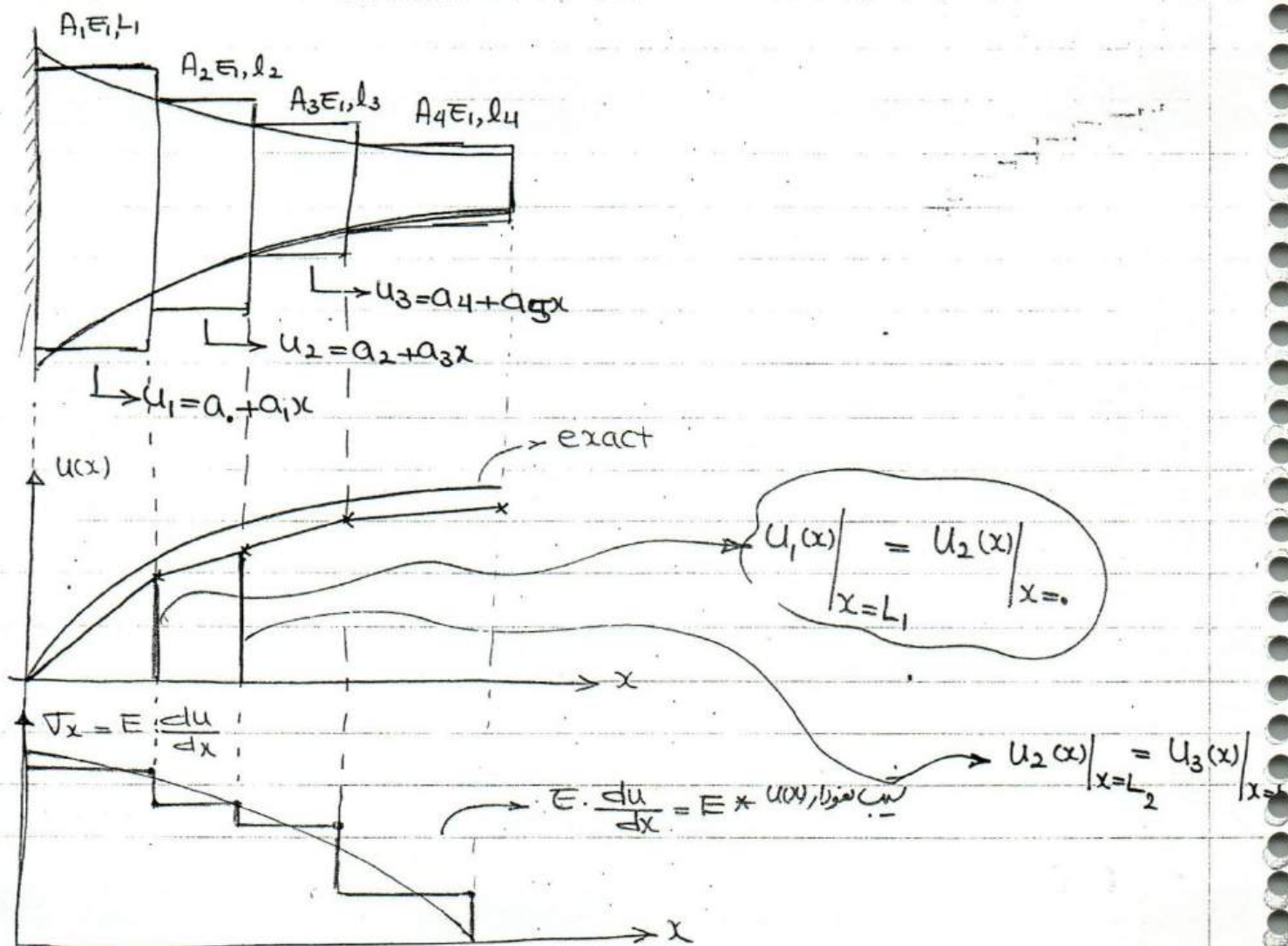
$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

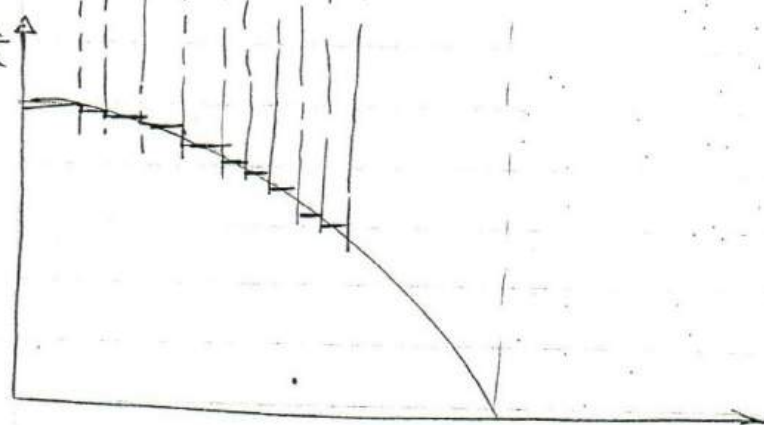
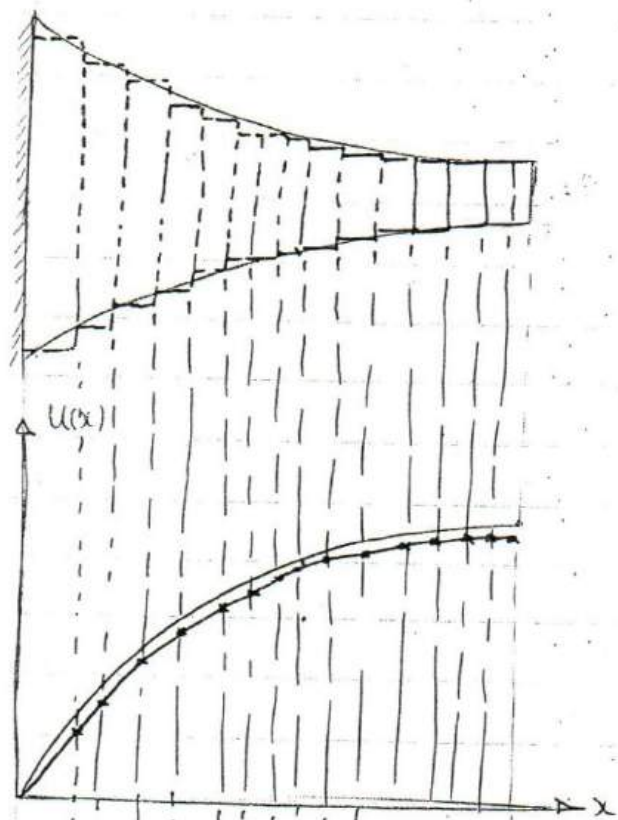
$$U = \int_L \frac{1}{2} (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3)^2 dx$$

$$\Rightarrow \pi = U + V$$

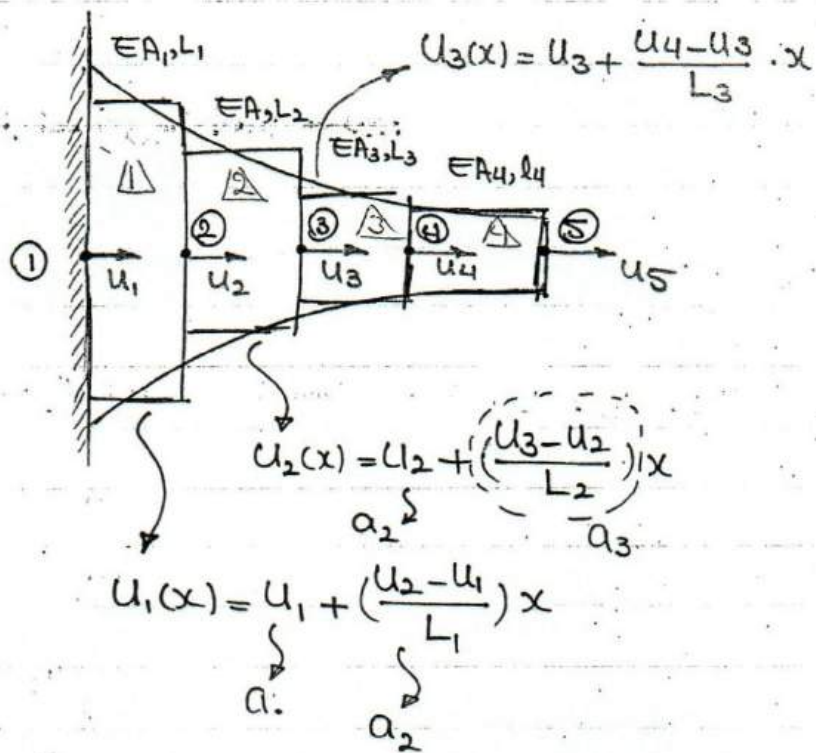
V = ...

استفاده از روش ریلی، ابتدا جایی حولانی و پیر هرنه اری





27.



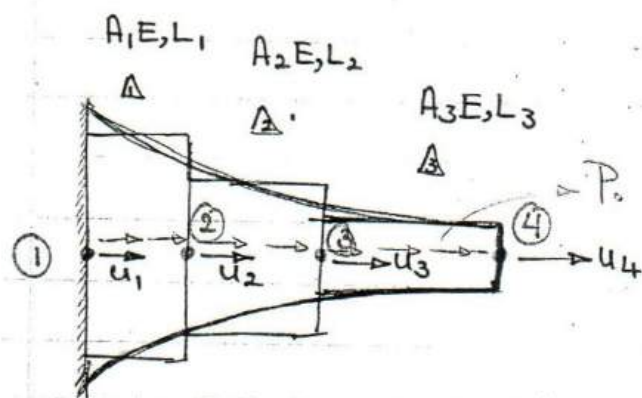
$$T(u_1, u_2, \dots) = U(u_1, u_2, \dots) + V(u_1, u_2, \dots)$$

$$\longrightarrow \delta\pi = . \longrightarrow u_1, u_2, \dots \rightsquigarrow [K] \underline{u} = \{F\}$$

$$u(x) = u_i + \frac{u_{i+1} - u_i}{L_i} x = \left(1 - \frac{x}{L_i}\right) u_i + \frac{x}{L_i} u_{i+1}$$

$$= \left\langle 1 - \frac{x}{L_i}, \frac{x}{L_i} \right\rangle \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

سازي يك بعدي باروش الما محدود و کاربرد روش ريلي رتيز رخصيه انرژي تيا نيل سينم:



اعمال روش الما محدود:

مولات ما $\bar{u}_i$ هاشند u_4, u_3, u_2, u_1

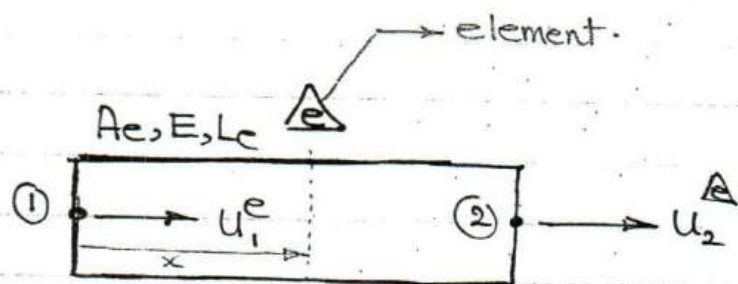
$$u_1(x) = \underbrace{u_1}_{a_0} + \underbrace{\frac{u_2 - u_1}{L_1}}_{a_1} x = \left(1 - \frac{x}{L_1}\right) u_1 + \frac{x}{L_1} u_2 = \left\langle 1 - \frac{x}{L_1} \quad \frac{x}{L_1} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u_2(x) = \underbrace{u_2}_{a_2} + \underbrace{\frac{u_3 - u_2}{L_2}}_{a_3} x = \left(1 - \frac{x}{L_2}\right) u_2 + \frac{x}{L_2} u_3 = \left\langle 1 - \frac{x}{L_2} \quad \frac{x}{L_2} \right\rangle \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_3(x) = \underbrace{u_3}_{a_4} + \underbrace{\frac{u_4 - u_3}{L_3}}_{a_5} x = \left(1 - \frac{x}{L_3}\right) u_3 + \frac{x}{L_3} u_4 = \left\langle 1 - \frac{x}{L_3} \quad \frac{x}{L_3} \right\rangle \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

x ها در مفاصل Local هاشند $\langle x_i < L_i \leftarrow$

اعمال روش ريلي - رتيز



$$u^e(x) = a_0 + a_1 x = u_1^e + \left(\frac{u_2^e - u_1^e}{L_e}\right) x$$

$$= \left(1 - \frac{x}{L_e}\right) u_1^e + \frac{x}{L_e} u_2^e$$

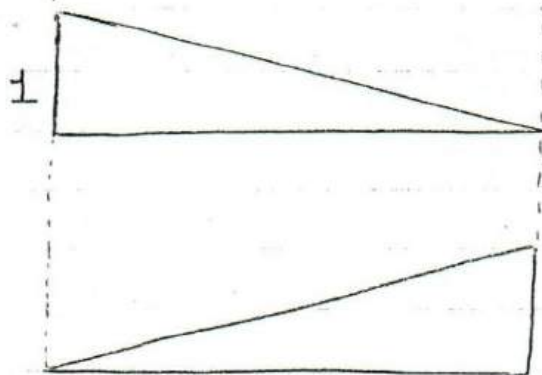
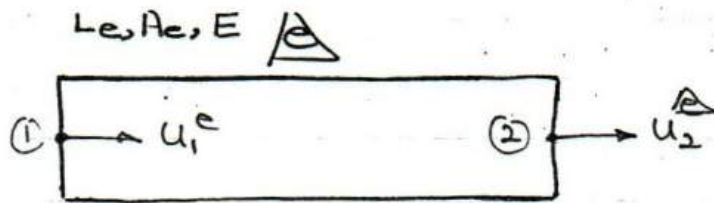
$$= \left\langle \underbrace{1 - \frac{x}{L_e}}_{N_1^e(x)} \quad \underbrace{\frac{x}{L_e}}_{N_2^e(x)} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

$\bar{u}_i^e$ معمول ها هاشند $\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$

$$u^e(x) = \left\langle 1 - \frac{x}{L_e} \quad \frac{x}{L_e} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \langle N_1^e(x) \quad N_2^e(x) \rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

قوابع شکل

Shape function.



$$\rightarrow N_1^e(x) = 1 - \frac{x}{L_e}$$

قوابع شکل ۱

$$\rightarrow N_2^e(x) = \frac{x}{L_e}$$

قوابع شکل ۲

خصوصیات قوابع شکلی : قابعی است که دارای مقدار واحد در هر مورد نظر و صفر در سایر نره ها باشند.

تعداد کل نره های المان $\rightarrow$ $\sum_{i=1}^{n_{node}} N_i^e(x) = 1$

Number of Node

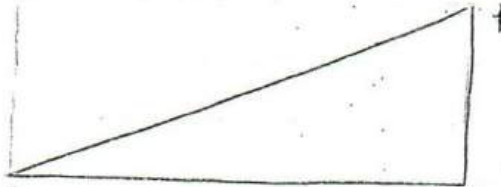
مجموع قوابع شکل تمام نره ها مقدار واحدی باشد.

$$u^e(x) = \langle N_1^e(x) \quad N_2^e(x) \rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \underline{N}^e(x) \underline{\bar{u}}^e$$

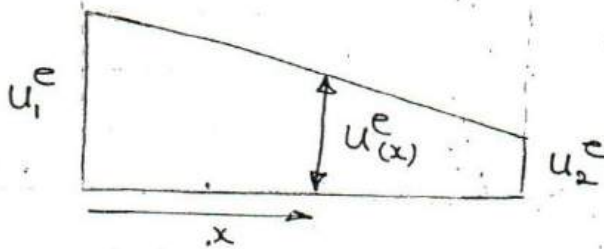
$$\rightarrow u^e(x) = \underline{N}^e(x) \underline{\bar{u}}^e \rightarrow \underline{\bar{u}}^e = \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$



$$[N_1^e(x) = (1 - \frac{x}{L_e})] \times u_1^e$$



$$[N_2^e(x) = \frac{x}{L_e}] \times u_2^e$$



$$u^e(x) = N_1(x) u_1^e + N_2(x) u_2^e$$

$$u(x) = [N_1(x) \quad N_2(x)] \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

درش المان e :

$$\epsilon_x^e = \frac{du^e}{dx} = \left\langle \frac{dN_1^e}{dx} \quad \frac{dN_2^e}{dx} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

B_1^e

B_2^e

مجموعات

$$= \left\langle -\frac{1}{L_e} \quad \frac{1}{L_e} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

$$= \left\langle B_{1(x)}^e \quad B_{2(x)}^e \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

هر تابع شکل مثل از سه ج باشد B می تواند تابعی از x باشد یا نه باشد. اینجا B می تواند تابعی از x هم باشد. اینجا B می تواند تابعی از x هم باشد.

29.

$$\bar{u}^e = \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

$$\epsilon_x^e = \langle B_1^e \quad B_2^e \rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = B^e(x) \bar{u}^e \rightarrow \epsilon_x^e = B^e(x) \cdot \bar{u}^e$$

3. اعمال قضیه انرژی پتانسیل مینم.

$$\Pi = U + V$$

$$\begin{cases} U = \sum u_i^A = u_1^A + u_2^A + u_3^A \\ V = \sum v_i^A = v_1^A + v_2^A + v_3^A \end{cases}$$

$$\triangle \Rightarrow U^A = \int_{V^e} \frac{1}{2} \sigma_{ij}^e \epsilon_{ij}^e dV = \int_{V^e} \frac{1}{2} \sigma_x^e \cdot \epsilon_x^e \cdot dV$$

$$= \int_{L^e} \frac{1}{2} A_e \cdot E \cdot \left(\frac{du^e}{dx} \right)^2 dx$$

$$(\bar{u}^e)^T B^e$$

$$(B^e \cdot \bar{u}^e)^2 = (B^e \bar{u}^e)^T \cdot (B^e \bar{u}^e)$$

$$= \int_L \frac{1}{2} A_e \cdot E \cdot (B^e \bar{u}^e)^T (B^e \bar{u}^e) dx$$

$$= \bar{u}^e{}^T \left(\int_{L^e} \frac{1}{2} A_e \cdot E \cdot B^e{}^T B^e \cdot dx \right) \cdot \bar{u}^e$$

معمولات $\bar{u}^e$ و $\bar{u}^e{}^T$ هستند.

$\bar{u}^e$ و $\bar{u}^e{}^T$ ثابت هستند از انحراف خارج می شوند

$$U^e = \langle u_1^e \ u_2^e \rangle \left[\frac{1}{L_e} A_e E \begin{Bmatrix} -\frac{1}{L_e} \\ \frac{1}{L_e} \end{Bmatrix} \langle -\frac{1}{L_e} \ \frac{1}{L_e} \rangle dx \right] \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

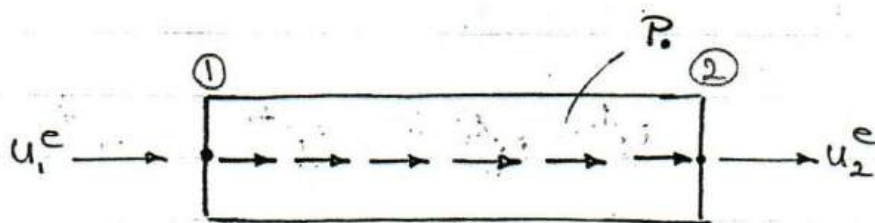
$$\begin{bmatrix} \frac{AE}{L_e} & -\frac{AE}{L_e} \\ -\frac{AE}{L_e} & \frac{AE}{L_e} \end{bmatrix} * L_e$$

$$= \frac{1}{2} \langle u_1^e \ u_2^e \rangle \cdot \begin{bmatrix} \frac{A_e E}{L_e} & -\frac{A_e E}{L_e} \\ -\frac{A_e E}{L_e} & \frac{A_e E}{L_e} \end{bmatrix} \cdot \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix}$$

$$U = \frac{1}{2} \bar{u}^T \cdot \underset{\sim}{k}^e \cdot \bar{u}^e$$

$$U^e = \frac{1}{2} \cdot \bar{u}^{eT} \cdot \underset{\sim}{k}^e \cdot \bar{u}^e$$

محصولات ما $\bar{u}^e$ ، $\bar{u}^{eT}$ هستند که ترنزشده اند



افزایش پتانسیل نیروهای خارجی

المان Δ

$$\Rightarrow V^{\Delta} = - \int_{L_e} P(x) u(x) dx$$

$(\underline{N}^e \alpha) \cdot \bar{u}^e)^T$

لاگرافیک اسکالر است همچنین $\bar{u}^e$ و $\underline{N}^e$
 چگونه می توان مقیوس کردن آن ها تا تیری در جواب ندارد

$$\Rightarrow V^{\Delta} = - \int_L P \cdot \underline{N}^e T dx$$

معمول

اصلاً چرا تراشیدنی لینم؟

چون بردار $\underline{N}^e$ ستونی است پس بردار
 $\bar{u}^e$ باید سطر باشد

$$= - \langle u_1^e \quad u_2^e \rangle \int_{L_e} P \cdot \left\{ \begin{array}{c} 1 - \frac{x}{L_e} \\ \frac{x}{L_e} \end{array} \right\} dx$$

$$= - \langle u_1^e \quad u_2^e \rangle \left\{ \begin{array}{c} \frac{P_0 L_e}{2} \\ \frac{P_0 L_e}{2} \end{array} \right\}$$

$$V^{\Delta} = - \bar{u}^e T \cdot \underline{f}^e$$

بردار نیروی المان

تایما $\bar{u}^e$ ، $\underline{N}^e T$ هستند

یعنی $\langle u_1^e \quad u_2^e \rangle$ ، $\left\{ \begin{array}{c} \frac{P_0 L_e}{2} \\ \frac{P_0 L_e}{2} \end{array} \right\}$

بهترین شده اند

$$\text{مجموعات} \rightarrow \bar{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

پارامترهایی که در فرم شده اند مجهولات هستند.

$$u = u^{\Delta} + u^{\Delta} + u^{\Delta} = \frac{1}{2} \bar{u}^{\Delta T} \hat{k}^{\Delta} \bar{u}^{\Delta} + \frac{1}{2} \bar{u}^{\Delta T} \hat{k}^{\Delta} \bar{u}^{\Delta} + \frac{1}{2} \bar{u}^{\Delta T} \hat{k}^{\Delta} \bar{u}^{\Delta}$$

از روی دل بستیم

$$= \frac{1}{2} \langle u_1 \ u_2 \rangle \begin{bmatrix} \frac{A_1 E}{L} & -\frac{A_1 E}{L_1} \\ -\frac{A_1 E}{L_1} & \frac{A_1 E}{L_1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$+ \frac{1}{2} \langle u_2 \ u_3 \rangle \begin{bmatrix} \frac{A_2 E}{L_2} & -\frac{A_2 E}{L_2} \\ -\frac{A_2 E}{L_2} & \frac{A_2 E}{L_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$+ \frac{1}{2} \langle u_3 \ u_4 \rangle \frac{A_3 E}{L_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{1}{2} \underbrace{\langle \bar{u}_1 \ \bar{u}_2 \ \bar{u}_3 \ \bar{u}_4 \rangle}_{\bar{u}^T} \underbrace{\begin{bmatrix} \frac{A_1 E}{L_1} & -\frac{A_1 E}{L_1} & 0 & 0 \\ -\frac{A_1 E}{L_1} & \frac{A_1 E}{L_1} & 0 & 0 \\ 0 & 0 & \frac{A_2 E}{L_2} & -\frac{A_2 E}{L_2} \\ 0 & 0 & -\frac{A_2 E}{L_2} & \frac{A_2 E}{L_2} \end{bmatrix}}_{\hat{k}^{\Delta}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\bar{u}}$$

$$+ \frac{1}{2} \langle u_1 \boxed{u_2 \ u_3} \ u_4 \rangle$$

$$\underbrace{\hspace{10em}}_{\tilde{u}^T}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{A_2 E}{L_2} & -\frac{A_2 E}{L_2} & 0 \\ 0 & -\frac{A_2 E}{L_2} & \frac{A_2 E}{L_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{K}^A} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\tilde{u}}$$

$$+ \frac{1}{2} \langle u_1 \ u_2 \ \boxed{u_3 \ u_4} \rangle$$

$$\underbrace{\hspace{10em}}_{\tilde{u}^T}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{A_3 E}{L_3} & -\frac{A_3 E}{L_3} \\ 0 & 0 & -\frac{A_3 E}{L_3} & \frac{A_3 E}{L_3} \end{bmatrix}}_{\hat{K}^B} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\tilde{u}}$$

$$= \frac{1}{2} \langle u_1 \ u_2 \ u_3 \ u_4 \rangle$$

$$\underbrace{\hspace{10em}}_{\tilde{u}^T}$$

$$\underbrace{\begin{bmatrix} \frac{A_1 E}{L_1} & -\frac{A_1 E}{L_1} & 0 & 0 \\ -\frac{A_1 E}{L_1} & \frac{A_1 E}{L_1} + \frac{A_2 E}{L_2} & -\frac{A_2 E}{L_2} & 0 \\ 0 & -\frac{A_2 E}{L_2} & \frac{A_2 E}{L_2} + \frac{A_3 E}{L_3} & -\frac{A_3 E}{L_3} \\ 0 & 0 & -\frac{A_3 E}{L_3} & \frac{A_3 E}{L_3} \end{bmatrix}}_{\text{Sym}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\tilde{u}}$$

$$= \frac{1}{2} \tilde{u}^T \hat{K} \tilde{u}$$

$$\hat{K} = \sum \hat{K}^A$$

← مَارِسِ سَفِي حَرَامَان

← مَارِسِ سَفِي حَرَامَان

فرضیه: یادگیری:

یها ← تغییر مکان های نره ای هستند و مقدارشان هم معمول هستند

$$V^e = -\bar{u}^e T \cdot \hat{p}^e = -\langle u_1^e \quad u_2^e \rangle \begin{Bmatrix} P_0 L e/2 \\ P_0 L e/2 \end{Bmatrix} = -\bar{u}^e T \cdot \hat{p}^e$$

$$V = \sum_e V^e = V^1 + V^2 + V^3$$

$$= -\bar{u}^1 T \cdot \hat{p}^1 + (-\bar{u}^2 T \cdot \hat{p}^2) - \bar{u}^3 T \cdot \hat{p}^3$$

$$= -[u_1 \quad u_2] \begin{Bmatrix} \frac{P_0 L_1}{2} \\ \frac{P_0 L_1}{2} \end{Bmatrix} - [u_2 \quad u_3] \begin{Bmatrix} \frac{P_0 L_2}{2} \\ \frac{P_0 L_2}{2} \end{Bmatrix} - [u_3 \quad u_4] \begin{Bmatrix} \frac{P_0 L_2}{2} \\ \frac{P_0 L_2}{2} \end{Bmatrix}$$

$$= -\underbrace{[u_1 \quad u_2 \quad u_3 \quad u_4]}_{\bar{u}^T} \begin{Bmatrix} \frac{P_0 L_1}{2} \\ \frac{P_0 L_1}{2} \\ \vdots \end{Bmatrix} \rightarrow \hat{p}^{\Delta}$$

$$- \underbrace{[u_1 \quad u_2 \quad u_3 \quad u_4]}_{\bar{u}^T} \begin{Bmatrix} \frac{P_0 L_2}{2} \\ \frac{P_0 L_2}{2} \\ \vdots \end{Bmatrix} \rightarrow \hat{p}^{\Delta}$$

←

$$\begin{aligned}
 & - \underbrace{[u_1 \ u_2 \ u_3 \ u_4]}_{\tilde{u}^T} \underbrace{\begin{Bmatrix} \vdots \\ \frac{P.L_3}{2} \\ \frac{P.L_3}{2} \end{Bmatrix}}_{\tilde{F}} \\
 & = - \underbrace{[u_1 \ u_2 \ u_3 \ u_4]}_{\tilde{u}^T} \underbrace{\begin{Bmatrix} \frac{P.L_1}{2} \\ \frac{P.L_1}{2} + \frac{P.L_2}{2} \\ \frac{P.L_2}{2} + \frac{P.L_3}{2} \\ \frac{P.L_3}{2} \end{Bmatrix}}_{\tilde{F}}
 \end{aligned}$$

$$V = \sum_e V^e = - \tilde{u}^T \tilde{F}$$

$\tilde{F} = \sum \tilde{F}^e$

بردار نیروی هر المان $\rightarrow$ بردار نیروی کل سیستم

$$\pi_{(g)} = U + V = \frac{1}{2} \cdot \tilde{u}^T \cdot K \cdot \tilde{u} - \tilde{u}^T \tilde{F}$$

$K^T = K$

مقارن است.

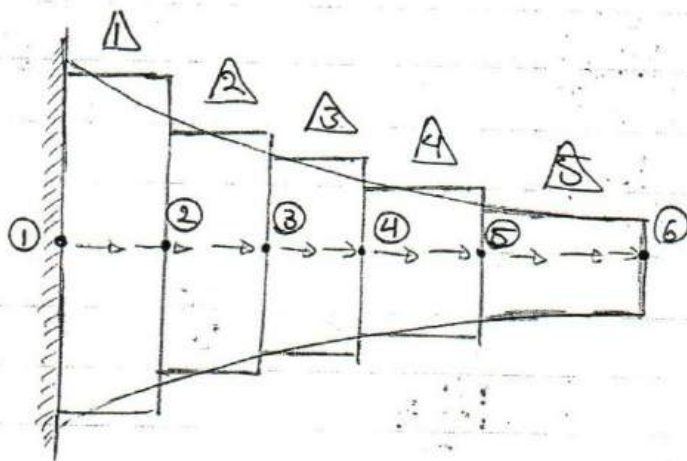
$$\delta \pi = \frac{1}{2} \cdot \delta \tilde{u}^T \cdot K \cdot \tilde{u} + \frac{1}{2} \left[\tilde{u}^T \cdot K \cdot \delta \tilde{u} \right] - \delta \tilde{u}^T \cdot \tilde{F} = 0$$

انرژی لابی اسکالر است پس می توان آن را آنسپوز کرد:

$$\delta \tilde{u}^T [K \tilde{u} - \tilde{F}] = 0 \rightarrow K \tilde{u} = \tilde{F}$$

$\delta \tilde{u}^T \neq 0$

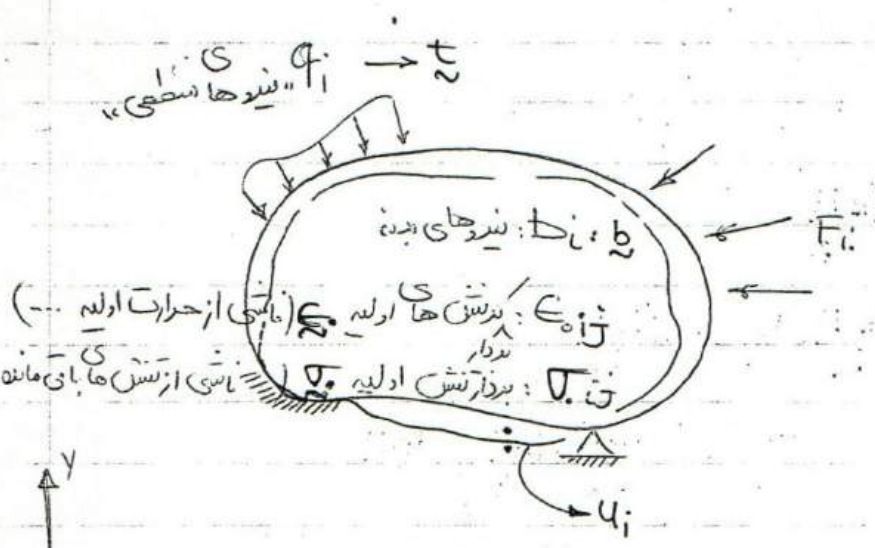
چرا؟



u_1 u_2 u_3 u_4 u_5 u_6

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

مدل سازی المان محدود مسائل محیط های پیوسته با کاربرد روش رانز و اعمال قضیه انرژی پتانسیل



$$\underline{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

$$\underline{t} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

$$\underline{\epsilon}_0 = \begin{Bmatrix} \epsilon_{0x} \\ \epsilon_{0y} \\ \epsilon_{0z} \\ \gamma_{0xy} \\ \gamma_{0yz} \\ \gamma_{0zx} \end{Bmatrix}$$

$$\underline{\sigma}_0 = \begin{Bmatrix} \sigma_{0x} \\ \sigma_{0y} \\ \sigma_{0z} \\ \tau_{0xy} \\ \tau_{0yz} \\ \tau_{0zx} \end{Bmatrix}$$

این ها از ابتدا مقدار دارند
صفرشان اونو از ابتدا مقدار دارند

u_1	u_2	u_3	u_4	u_5	u_6
$\frac{A_1 E}{L_1}$	$-\frac{A_1 E}{L_1}$				
$-\frac{A_1 E}{L_1}$	$\frac{A_1 E}{L_1} + \frac{A_2 E}{L_2}$	$-\frac{A_2 E}{L_2}$			
	$-\frac{A_2 E}{L_2}$	$\frac{A_2 E}{L_2} + \frac{A_3 E}{L_3}$	$-\frac{A_3 E}{L_3}$		
		$-\frac{A_3 E}{L_3}$	$\frac{A_3 E}{L_3} + \frac{A_4 E}{L_4}$	$-\frac{A_4 E}{L_4}$	
			$-\frac{A_4 E}{L_4}$	$\frac{A_4 E}{L_4} + \frac{A_5 E}{L_5}$	$-\frac{A_5 E}{L_5}$
				$-\frac{A_5 E}{L_5}$	$\frac{A_5 E}{L_5}$

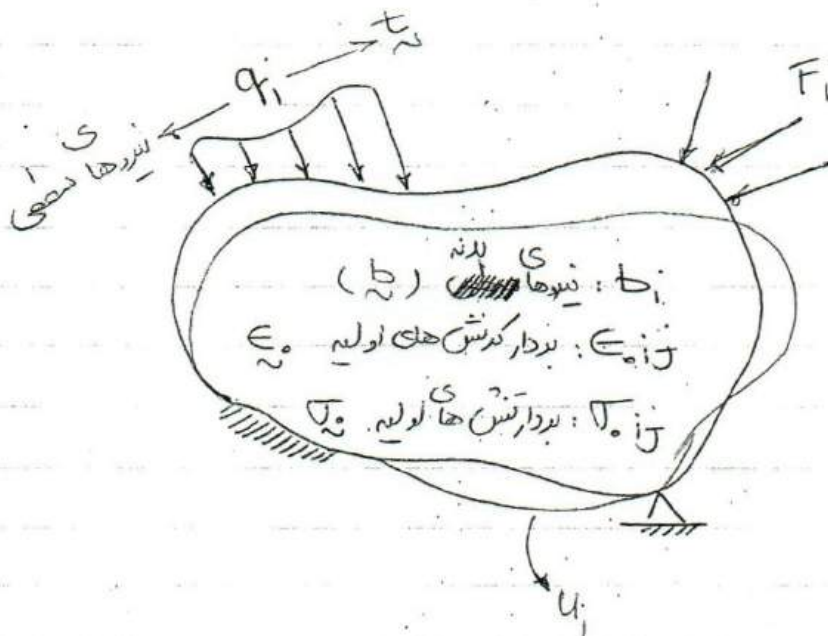
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} \frac{P_0 L_1}{2} \\ \frac{P_0 L_1}{2} + \frac{P_0 L_2}{2} \\ \frac{P_0 L_2}{2} + \frac{P_0 L_3}{2} \\ \frac{P_0 L_3}{2} + \frac{P_0 L_4}{2} \\ \frac{P_0 L_4}{2} + \frac{P_0 L_5}{2} \\ \frac{P_0 L_5}{2} \end{Bmatrix}$$

$$\rightarrow u_1 = 0$$

$\det K = 0 \rightarrow$ ماتریس معکوس ندارد $\rightarrow$ محموله؛ و تقابل دارد $\Rightarrow$ Rigid body

$$\sum K \bar{u} = \sum F \xrightarrow[\bar{u}_1=0]{\text{B.C. اعمال}} \sum \hat{K} \hat{u} = \sum \hat{F}$$

تعدادی اعلان محدود مسائل محلی نوشته جا کاربرد روش ریلی ریزد اعمال قضیه انرژی پتانسیل مینیمم:

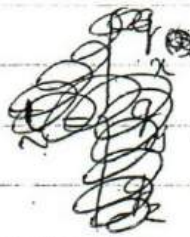


ϵ_0 : بردار کرنش های اولیه ϵ_0 « ناشی از حرارت و ... » مقدار معلوم دارند

σ_0 : بردار تنش های اولیه σ_0 « ناشی از تنش های باقی مانده در جسم ... » مقدار معلوم دارند. Residual stress

ϵ_0 و σ_0 از ابتدا حل دارای مقدار هستند و به همین علت آنها را با اندیس ۰ نشانی دهیم که
یعنی نموده اند که شود که از ابتدا مقدار دارند xxx

$$b = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$



$$t = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

$$\epsilon_0 = \begin{Bmatrix} \epsilon_{0x} \\ \epsilon_{0y} \\ \epsilon_{0z} \\ \gamma_{0xy} \\ \gamma_{0yz} \\ \gamma_{0zx} \end{Bmatrix}$$

$$\sigma_0 = \begin{Bmatrix} \sigma_{0x} \\ \sigma_{0y} \\ \sigma_{0z} \\ \tau_{0xy} \\ \tau_{0yz} \\ \tau_{0zx} \end{Bmatrix}$$

$u_1 \leftarrow \uparrow$
 $\text{delete } \uparrow$
 $u_1 = 0$

	u_1	u_2	u_3	u_4	u_5	u_6
$\frac{A_1 E}{L_1}$	$-\frac{A_1 E}{L_1}$	0	0	0	0	0
$-\frac{A_1 E}{L_1}$	$\frac{A_1 E}{L_1} + \frac{A_2 E}{L_2}$	$-\frac{A_2 E}{L_2}$	0	0	0	0
0	$-\frac{A_2 E}{L_2}$	$\frac{A_2 E}{L_2} + \frac{A_3 E}{L_3}$	$-\frac{A_3 E}{L_3}$	0	0	0
0	0	$-\frac{A_3 E}{L_3}$	$\frac{A_3 E}{L_3} + \frac{A_4 E}{L_4}$	$-\frac{A_4 E}{L_4}$	0	0
0	0	0	$-\frac{A_4 E}{L_4}$	$\frac{A_4 E}{L_4} + \frac{A_5 E}{L_5}$	$-\frac{A_5 E}{L_5}$	0
0	0	0	0	$-\frac{A_5 E}{L_5}$	$\frac{A_5 E}{L_5}$	0

$\left[\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} \right] = \left[\begin{matrix} \frac{P_0 L_1}{2} \\ \frac{P_0 L_1}{2} + \frac{P_0 L_2}{2} \\ \frac{P_0 L_2}{2} + \frac{P_0 L_3}{2} \\ \frac{P_0 L_3}{2} + \frac{P_0 L_4}{2} \\ \frac{P_0 L_4}{2} + \frac{P_0 L_5}{2} \\ \frac{P_0 L_5}{2} \end{matrix} \right]$

$K \vec{u} = \vec{F} \rightarrow \det(K) = 0 \Rightarrow$ این ماتریس معکوس ندارد $\Rightarrow$ حوضه آزاد ندارد $\Rightarrow$ Rigid motion دارد.

$u_1 = 0$ با اعمال شرایط مرزی $\xrightarrow[u_1=0]{BC}$ $\hat{K} \vec{u} = \hat{F}$

$$\Pi = U + V = \int_V u_0 dV - \int_S t_i u_i ds - \int_V b_i u_i dV$$

$$\delta \Pi = 0 \Rightarrow \int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_S t_i \delta u_i ds - \int_V b_i \delta u_i dV = 0$$

چرا $\frac{1}{2}$ ندارد ؟

$$U = \int u_0 dV = \int \frac{1}{2} \sigma_{ij} \epsilon_{ij} = \int \frac{1}{2} E \epsilon_{ij}^2 dV$$

$$\delta U = \int \frac{1}{2} \cancel{E} \cancel{\epsilon_{ij}} \delta \epsilon_{ij} dV = \int \sigma_{ij} \delta \epsilon_{ij} dV$$

چرا σ_{ij} معکول است ؟

$$\sigma_{ij} = E \epsilon_{ij} \longrightarrow \epsilon_{ij} \text{ معکول است}$$

$$\sigma_{ij} = D \cdot \epsilon_{ij}$$

بنابراین σ_{ij} هم معکول است

No.

PAN

$$\underline{\sigma} = \underline{D} (\underline{\epsilon} - \underline{\epsilon}_0) + \underline{\sigma}_0$$

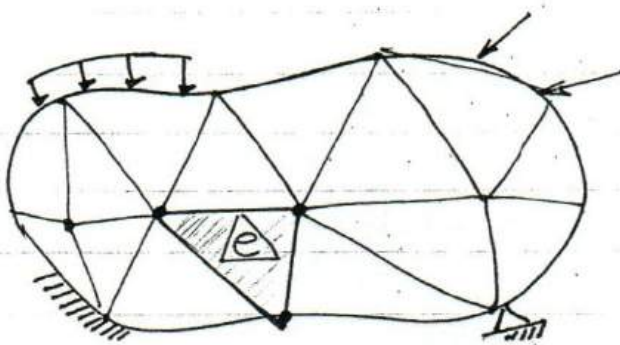
Review:

$$\Pi = u + v = \int_V \frac{1}{2} \cdot \sigma_{ij} \epsilon_{ij} dV - \left\{ \int_S t_i u_i ds + \int_V b_i u_i dV \right\}$$

$$\delta \Pi = 0 \rightarrow \int \sigma_{ij} \delta \epsilon_{ij} dV - \int t_i \delta u_i ds - \int b_i \delta u_i dV = 0$$

صورت برداری و ماتریسی :

$$\int \delta \underline{\epsilon}^T \cdot \underline{\sigma} \cdot dV - \int \delta \underline{u}^T \cdot \underline{t} \cdot ds - \int \delta \underline{u}^T \cdot \underline{b} \cdot dV = 0 \quad (I)$$



$$\sum_e \int_{V_e} \delta \underline{\epsilon}^T \cdot \underline{\sigma}^e \cdot dV - \sum_e \int_{S_e} \delta \underline{u}^T \cdot \underline{t}^e \cdot ds - \sum_e \int_{V_e} \delta \underline{u}^T \cdot \underline{b}^e \cdot dV = 0$$

۱- قضیه انرژی پتانسیل میسریم:

$$\Pi = U + V = \int_V u_0 dV - \int_S t_i u_i ds - \int_V b_i u_i dV$$

$$\delta \Pi = 0 \rightarrow \int_V \underbrace{\sigma_{ij}}_{6 \times 1} \underbrace{\delta \epsilon_{ij}}_{6 \times 1} dV - \int_S \underbrace{t_i}_{3 \times 1} \underbrace{\delta u_i}_{3 \times 1} ds - \int_V \underbrace{b_i}_{3 \times 1} \underbrace{\delta u_i}_{3 \times 1} dV = 0$$

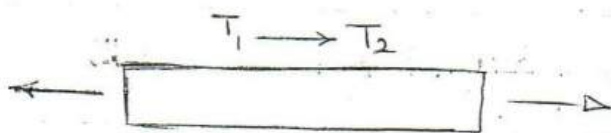
در حالت 2D

یادرمین برداری:

$$\int_V \underbrace{\delta \underline{\epsilon}}_{1 \times 6}^T \underbrace{\underline{\sigma}}_{6 \times 1} dV - \int_S \underbrace{\delta \underline{u}}_{1 \times 3}^T \underbrace{\underline{t}}_{3 \times 1} ds - \int_V \underbrace{\delta \underline{u}}_{1 \times 3}^T \underbrace{\underline{b}}_{3 \times 1} dV = 0$$

** دروش اولیه و تنش اولیه را باید در یک درجی بنیم **

معادلات با ترمز مشخص شده اند.



$$\epsilon_{ox} = \alpha \Delta T$$

$$\sigma = E(\epsilon - \epsilon_{ox})$$

$$\underline{\sigma} = \underline{D} \cdot \underline{\epsilon} \quad \text{تابعی از } \underline{\epsilon} \text{ و } \underline{D} \Rightarrow \text{پارسی خواص ماده}$$

$$\underline{\sigma} = \underline{D} \cdot \underline{\epsilon}$$

$$\text{در صورت وجود تنش اولیه} \Rightarrow \underline{\sigma} = \underline{D} (\underline{\epsilon} - \underline{\epsilon}_0)$$

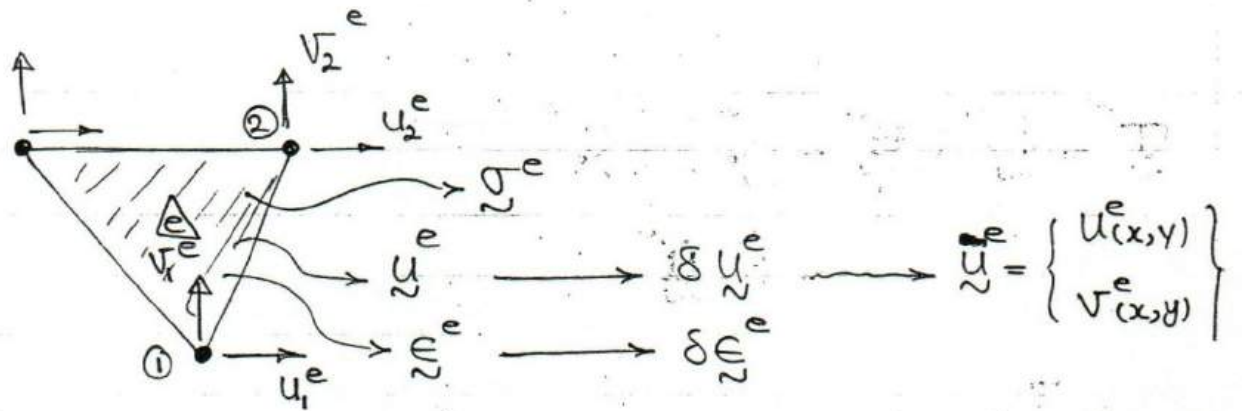
$\underline{\epsilon}_0$

$$\text{در صورتی که علاوه بر تنش اولیه تنش اولیه} \Rightarrow (\underline{\sigma} - \underline{\sigma}_0) = \underline{D} (\underline{\epsilon} - \underline{\epsilon}_0)$$

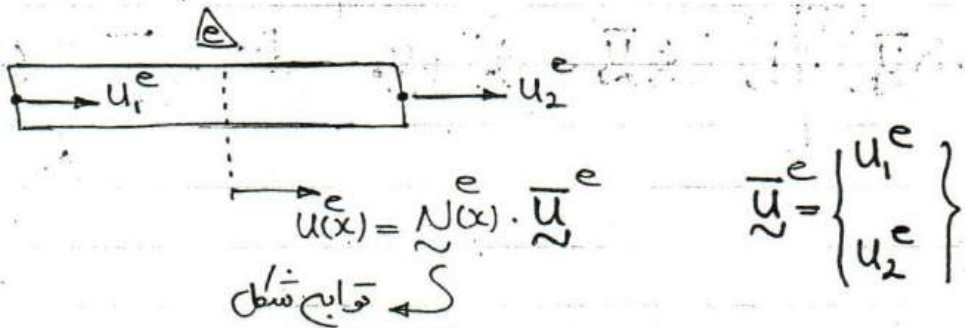
$\underline{\sigma}_0$ نیز وجود داشته باشد

$\underline{\sigma}$ و $\underline{\epsilon}$ معمول هستند که قویتر شده اند.

35.



1.D حالة :



$$z^e = z^e \cdot z1^e \longrightarrow \delta z^e = z^e \cdot \delta z1^e \longrightarrow \text{????}$$

$$z^e = \frac{dz^e}{dx} \cdot z1^e = B^e \cdot z1^e$$

$$z^e = B^e \cdot z1^e \longrightarrow \delta z^e = B^e \cdot \delta z1^e$$

$$\sigma = D \cdot \epsilon \longrightarrow (\sigma - \sigma_0) = D \cdot (\epsilon - \epsilon_0)$$

$$(\sigma - \sigma_0) = D (\epsilon - \epsilon_0)$$

$$\Rightarrow \sigma^e = D^e (\epsilon^e - \epsilon_0^e) + \sigma_0^e$$

$\downarrow$
 $B^e \cdot z1^e$

$$\underline{\sigma}^e = \underline{D}^e (\underline{\epsilon}^e - \underline{\epsilon}_0^e) + \underline{\sigma}_0^e$$

$\downarrow$
 $\underline{B}^e \underline{\bar{u}}^e$

نیازی $\underline{u}^e$ ، $\underline{\epsilon}^e$ ، $\underline{\sigma}^e$ در رابطه نداریم:

$$\sum_e \int_{V_e} (\underline{B}^e \delta \underline{\bar{u}}^e)^T [\underline{D}^e (\underline{B}^e \underline{\bar{u}}^e - \underline{\epsilon}_0^e) + \underline{\sigma}_0^e] dV - \sum_e \int_{S_e} (\underline{N}^e \delta \underline{\bar{u}}^e)^T \underline{t}^e dS$$

$\uparrow$ $\delta \underline{\bar{u}}^{eT} \underline{B}^{eT}$

$\downarrow$
 $\delta \underline{\bar{u}}^{eT} \underline{N}^{eT}$

$$- \sum_e \int_{V_e} (\underline{N}^e \delta \underline{\bar{u}}^e)^T \underline{b}^e dV = 0$$

$\delta \underline{\bar{u}}^{eT} \underline{N}^{eT}$

$$\sum_e \int_{V_e} \delta \underline{\bar{u}}^{eT} \underline{B}^{eT} \cdot \underline{D}^e \cdot \underline{B}^e \underline{\bar{u}}^e dV - \sum_e \int_{V_e} \delta \underline{\bar{u}}^{eT} \underline{B}^{eT} \cdot \underline{D}^e \underline{\epsilon}_0^e dV$$

$$+ \sum_e \int_{V_e} \delta \underline{\bar{u}}^{eT} \underline{B}^{eT} \cdot \underline{\sigma}_0^e dV - \sum_e \int_{S_e} \delta \underline{\bar{u}}^{eT} \underline{N}^{eT} \underline{t}^e dS$$

$$- \sum_e \int_{V_e} \delta \underline{\bar{u}}^{eT} \underline{N}^{eT} \underline{b}^e dV = 0 \quad (2)$$

تبدیل سیستم محلی (Local) به سیستم کلی (Global) داریم:

$\bar{u}^e$ ← درجات آزادی جابجایی المان
 $\bar{u}$ ← درجات آزادی جابجایی کل سیستم

$$\rightarrow \bar{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \quad \delta \bar{u} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta u_n \end{Bmatrix}$$

$$\sum_e \delta \bar{u}^T \hat{k}_e \bar{u} - \sum_e \delta \bar{u}^T \hat{f}_e^e - \sum_e \delta \bar{u}^T \hat{f}_\sigma^e - \sum_e \delta \bar{u}^T \hat{f}_t^e - \sum_e \delta \bar{u}^T \hat{f}_b^e = 0$$

$$\delta \bar{u}^T \left\{ \underbrace{\sum_e \hat{k}_e \bar{u}}_K - \underbrace{\sum_e \hat{f}_e^e}_{F_e} - \underbrace{\sum_e \hat{f}_\sigma^e}_{F_\sigma} - \underbrace{\sum_e \hat{f}_t^e}_{F_t} - \underbrace{\sum_e \hat{f}_b^e}_{F_b} \right\} = 0$$

$$K \cdot \bar{u} = F_e + F_\sigma + F_t + F_b$$

$$\rightarrow K \cdot \bar{u} = F$$

$$K^e = \int_{V_e} B^e{}^T D^e B^e dV \longrightarrow \text{ماتریس سختی المان}$$

$$F_{\epsilon}^e = \int_{V_e} B^e{}^T D^e \epsilon^e dV \longrightarrow \text{بردار معادل نیروهای گره‌ای ناشی از کرنش اولیه} \epsilon_0$$

$$F_{\sigma_0}^e = \int_{V_e} B^e{}^T \sigma_0^e dV \longrightarrow \text{بردار معادل نیروهای گره‌ای ناشی از تنش اولیه} \sigma_0$$

$$F_t^e = \int_{\Omega} N^e{}^T t^e d\Omega \longrightarrow \text{بردار معادل نیروهای گره‌ای ناشی از بارهای سطحی}$$

$$\rightarrow F_t^e = \int_{\Omega} N^e{}^T q^e d\Omega + F_c^e$$

C: Concentrate $\rightarrow$ بارها متمرکز گره

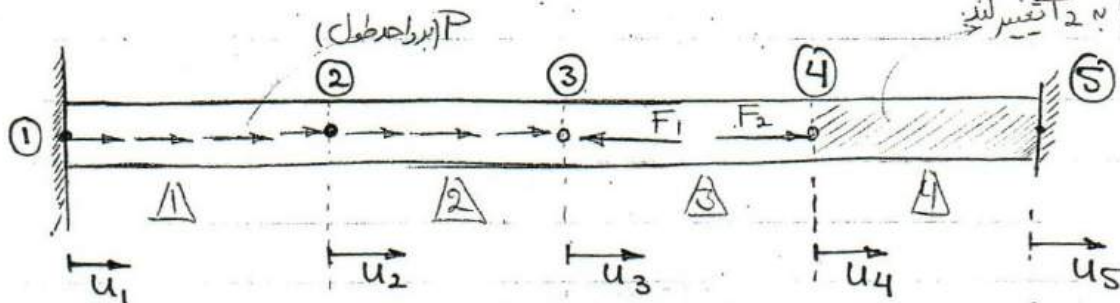
$$F_b^e = \int_{V_e} N^e{}^T b^e dV \longrightarrow \text{بردار معادل نیروهای گره‌ای ناشی از نیروهای بدنه}$$

$$\begin{aligned} (2) \rightarrow & \sum_e \delta \bar{u}^e{}^T K^e \bar{u}^e - \sum_e \delta \bar{u}^e{}^T F_{\epsilon}^e - \sum_e \delta \bar{u}^e{}^T F_{\sigma_0}^e \\ & - \sum_e \delta \bar{u}^e{}^T F_t^e - \sum_e \delta \bar{u}^e{}^T F_b^e = 0 \end{aligned}$$

$$\rightarrow \underline{\tilde{K}} \cdot \underline{\tilde{U}} = \underline{\tilde{F}} \rightarrow \underline{\tilde{F}} = \underbrace{\underline{\tilde{F}}_t}_{\text{Total}} + \underline{\tilde{F}}_c + \underline{\tilde{F}}_q + \underline{\tilde{F}}_b + \underline{\tilde{F}}_e + \underline{\tilde{F}}_T$$

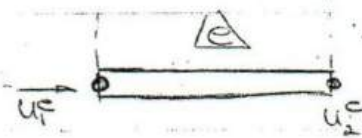
مثال: مطلوب است حل عضو محوری شکل زیر با استفاده از ایلان ها (دوگانه): $EA = \text{constant}$.

درجه حرارت از T_1 به T_2 تغییر کند



مقاومت ایلان ها: AE, L

α : ضریب انبساط حرارتی



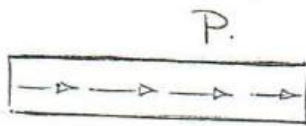
$$\begin{cases} N_1^e(x) = 1 - \frac{x}{L_e} \\ N_2^e = \frac{x}{L_e} \end{cases}$$

$$u(x) = \left\langle 1 - \frac{x}{L_e}, \frac{x}{L_e} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \underline{\tilde{N}}^e(x) \cdot \underline{\tilde{U}}^e$$

$$\underline{\epsilon}_x^e = \left\langle -\frac{1}{L_e}, \frac{1}{L_e} \right\rangle \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \underline{\tilde{B}}^e(x) \underline{\tilde{U}}^e$$

$$\underline{\tilde{K}} = \int_V \underline{\tilde{B}}^T \cdot D \cdot \underline{\tilde{B}} \cdot dV = \int_L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \cdot E \cdot \left\langle \frac{1}{L} \quad -\frac{1}{L} \right\rangle \cdot A \cdot dx$$

$$\underline{\tilde{K}} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\underline{F}_q = \int_S \underline{N}^T \cdot \underline{q} \cdot dS = \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \cdot P \cdot dx = \begin{Bmatrix} \frac{PL}{2} \\ \frac{PL}{2} \end{Bmatrix} = \underline{F}_q^{\Delta} = \underline{F}_q^{\Delta}$$

$$\underline{F}_{e_0}^{\Delta} = \int B^T \cdot D \cdot \underline{\epsilon}_0 \cdot dV = \int_L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \cdot E \cdot \alpha \Delta T \cdot A \cdot dx = \begin{Bmatrix} -AE\alpha\Delta T \\ AE\alpha\Delta T \end{Bmatrix}$$

$\underline{\epsilon}_{0,x} = \alpha \Delta T$
 $= \alpha (T_2 - T_1)$

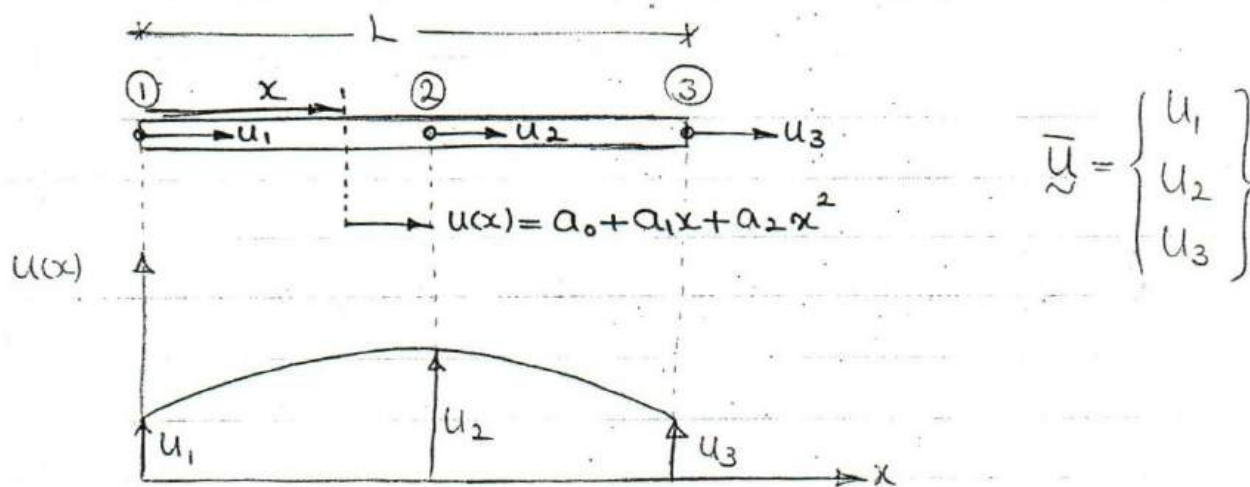
	u_1	u_2	u_3	u_4	u_5		$\underline{F}_e$
	1	-1	0	0	0	$\left\{ \begin{array}{c} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \right\} =$	0
$\frac{AE}{L}$	-1	1+1	-1	0	0		0
	0	-1	1+1	-1	0		-F_1
	0	0	-1	1+1	-1		F_2
	0	0	0	-1	1		0
	$\underline{F}_q \quad \quad \quad \underline{F}_{e_0}$						

$$+ \begin{Bmatrix} \frac{PL}{2} \\ \frac{PL}{2} + \frac{PL}{2} \\ \frac{PL}{2} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -AE\alpha\Delta T \\ AE\alpha\Delta T \end{Bmatrix}$$

$$\Rightarrow \begin{matrix} \hat{K} \\ \downarrow \\ 3 \times 3 \end{matrix} \cdot \begin{matrix} \hat{U} \\ \downarrow \\ 3 \times 1 \end{matrix} = \begin{matrix} \hat{F} \\ \downarrow \\ 3 \times 1 \end{matrix}$$

المان معوری سه گانه:

توجه: يك المان سه گانه است. «دوتا المان دو گانه ليست»



مستقيم تعيين تابع شكل:

Direct procedure

1- تعيين چند جمله ای درجه ۲ - رتبه ۱

$$u(x) = \begin{Bmatrix} 1 & x & x^2 \end{Bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \tilde{P}(x) \cdot \tilde{a} \quad \text{①}$$

بردار ضرایب درجه ۲ - رتبه ۱

بردار چند جمله ای ها.

۲- اعمال شرایط مرزی درجات آزادی جابه جایی:

$$u(x) \Big|_{x=0} = u_1 \rightarrow a_0 + 0 + 0 = u_1$$

$$u(x) \Big|_{x=\frac{L}{2}} = u_2 \rightarrow a_0 + a_1\left(\frac{L}{2}\right) + a_2\left(\frac{L^2}{4}\right) = u_2$$

$$u(x) \Big|_{x=L} = u_3 \rightarrow a_0 + a_1L + a_2L^2 = u_3$$

است.

درجه ها $(\bar{u}_i)$

با جابه جایی درجه ها

«میران جابه جایی»

هدف بردار کردن و ادغام بین تابع تغییر شکل

هدف

$$\begin{cases} u(x) = P(x) \cdot a \\ C \cdot a = \bar{u} \end{cases}$$

هدف این است که $u(x)$ را به صورت $N(x) \cdot \bar{u}$ بنویسیم
 در $u(x)$ را بر اساس جابه جایی کرده ایم

$$u(x) = N(x) \cdot \bar{u}$$

$$u(x) = N(x) \cdot \bar{u}$$

$$u(x) \rightarrow \bar{u}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \Rightarrow C \cdot a = \bar{u} \quad (1)$$

ماتریس ضرایب که درجات آزادی دلی و نیز درجه آزادی جابه جایی دارند

$$C \cdot a = \bar{u} \Rightarrow a = C^{-1} \cdot \bar{u} \quad (2)$$

از روابط (1) و (2)

$$u(x) = P(x) \cdot a$$

$$= \underbrace{P(x) \cdot C^{-1}}_{N(x)} \cdot \bar{u} = \langle N_1(x) \quad N_2(x) \quad N_3(x) \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \sum_{i=1}^3 N_i(x) \cdot u_i$$

$$N(x) = P(x) \cdot C_{3 \times 3}^{-1} = \langle 1 \quad x \quad x^2 \rangle \begin{bmatrix} 1 & 0 & 0 \\ -3/L & 4/L & -1/L \\ 2/L^2 & -4/L^2 & 2/L^2 \end{bmatrix}$$

$$N_1(x) = a_0 + a_1 x_0 + a_2 x_0 = 1$$

$$N_1(x) = a_0 + a_1 \times \frac{L}{2} + a_2 \times \frac{L^2}{4} = 0$$

$$N_1(x) = a_0 + a_1 \times L + a_2 \times L^2 = 0$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & L/2 & L^2/4 \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow N_1(x)$$

$$N_2(x) = a_0 + a_1 x_0 + a_2 x_0 = 0$$

$$N_2(x) = a_0 + a_1 \times \frac{L}{2} + a_2 \times \frac{L^2}{4} = 1$$

$$N_2(x) = a_0 + a_1 \times L + a_2 \times L^2 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L/2 & L^2/4 \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \Rightarrow N_2(x)$$

$$N_3(x) = a_0 + a_1 x_0 + a_2 x_0 = 0$$

$$N_3(x) = a_0 + a_1 \times \frac{L}{2} + a_2 \times \frac{L^2}{4} = 0$$

$$N_3(x) = a_0 + a_1 \times L + a_2 \times L^2 = 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & L/2 & L^2/4 \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow N_3(x)$$

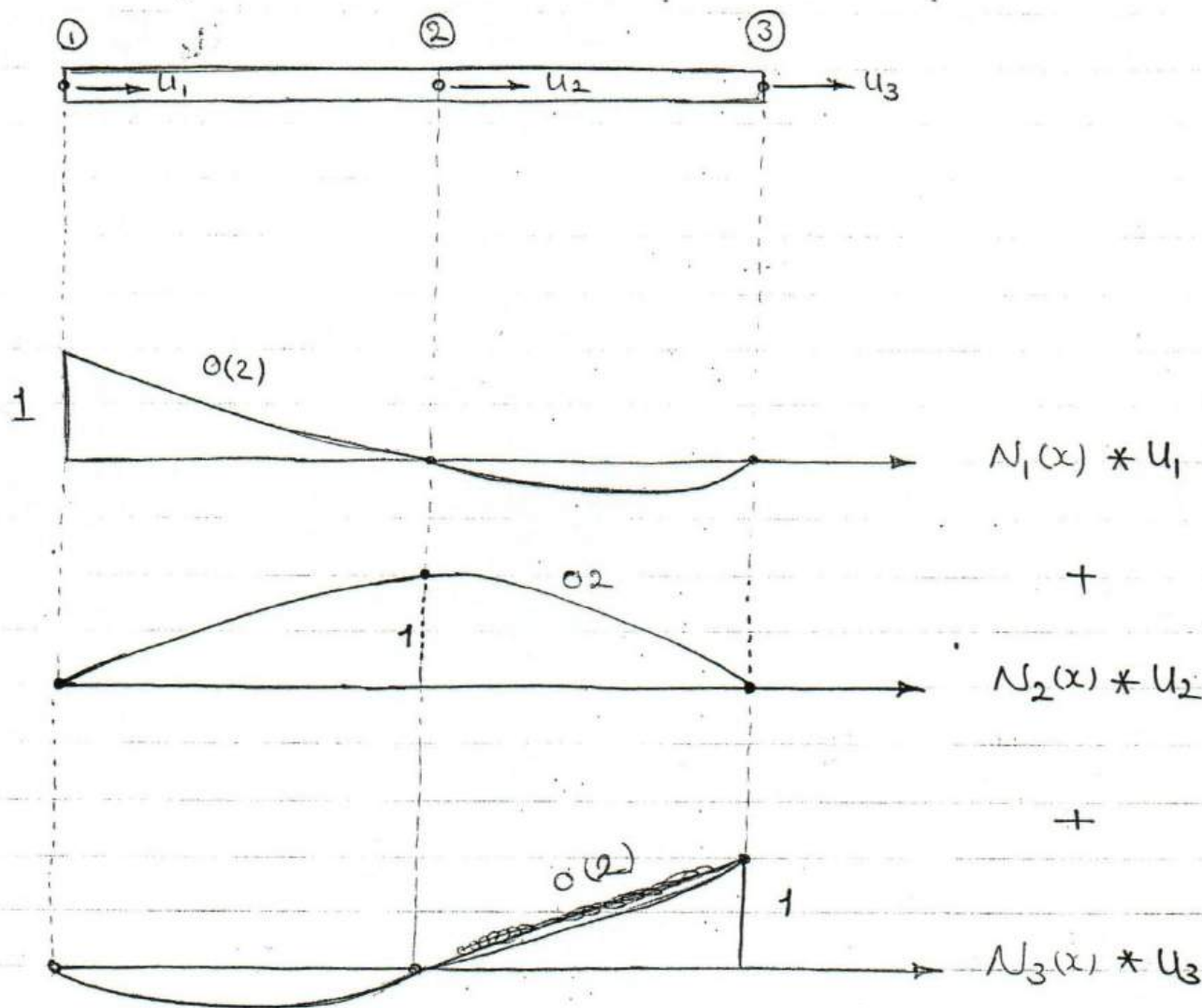
39.

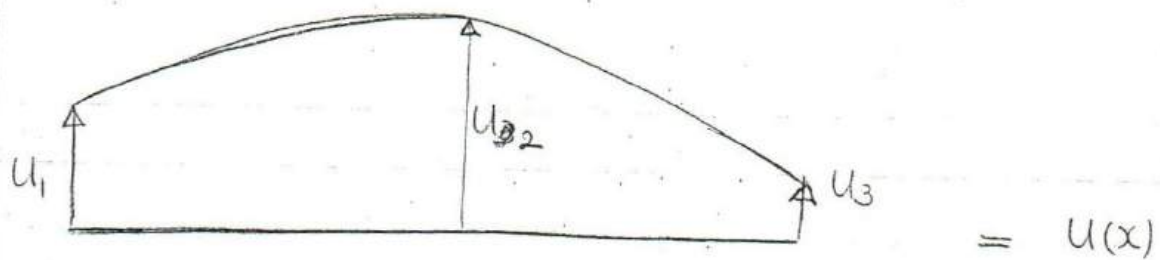
$$N(x) = \left\langle \underbrace{1 - \frac{3}{L}x + \frac{2}{L^2}x^2}_{N_1(x)} \quad \underbrace{\frac{4}{L}x - \frac{4}{L^2}x^2}_{N_2(x)} \quad \underbrace{-\frac{1}{L}x + \frac{2}{L^2}x^2}_{N_3(x)} \right\rangle$$

$$\sum_{i=1}^3 N_i = 1 \rightarrow \text{OK.}$$

$$\sum_{i=1}^3 N_i(x) = 1$$

Partition of unity.





$$\epsilon_x = \frac{du}{dx} = \left\langle \underbrace{\frac{dN_1}{dx}}_{B_1(x)} \quad \underbrace{\frac{dN_2}{dx}}_{B_2(x)} \quad \underbrace{\frac{dN_3}{dx}}_{B_3(x)} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \underline{\underline{B}} \cdot \underline{\underline{u}}$$

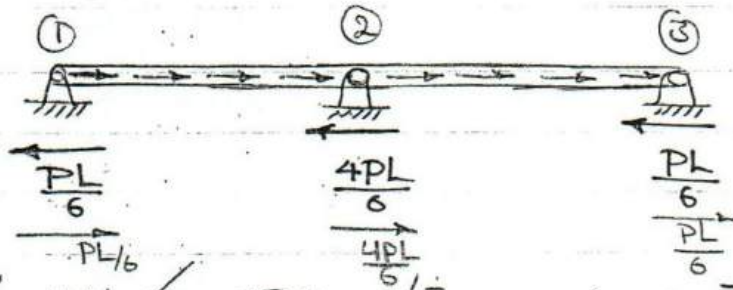
$$\epsilon_x = \left\langle -\frac{3}{L} + \frac{4}{L^2}x \quad \frac{4}{L} - \frac{8}{L^2}x \quad -\frac{1}{L} + \frac{4x}{L^2} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\underline{\underline{K}} = \int \underline{\underline{B}}^T \underline{\underline{D}} \underline{\underline{B}} \cdot dV = \int \begin{Bmatrix} -\frac{3}{L} + \frac{4}{L^2}x \\ \frac{4}{L} - \frac{8}{L^2}x \\ -\frac{1}{L} + \frac{4x}{L^2} \end{Bmatrix} \cdot E \cdot \left\langle -\frac{3}{L} + \frac{4}{L^2}x \quad \frac{4}{L} - \frac{8}{L^2}x \quad -\frac{1}{L} + \frac{4x}{L^2} \right\rangle \cdot dx$$

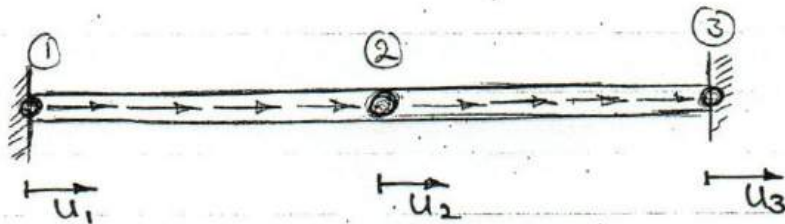
$$-\frac{1}{L} + \frac{4x}{L^2} \cdot A \cdot dx$$

$$\rightarrow \underline{\underline{K}} = \begin{matrix} & \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} & \begin{bmatrix} \frac{7}{3} & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix} \end{matrix} \cdot \frac{EA}{L}$$

آالفین مارتینی



①: مطلوب است حل عضو محوری مثل زیر با استفاده از یک المان سه نره:



$$\frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

$$\frac{AE}{3L} (16) \cdot u_2 = \frac{PL}{6} (4) \rightarrow u_2 = \frac{P \cdot L^2}{8AE}$$

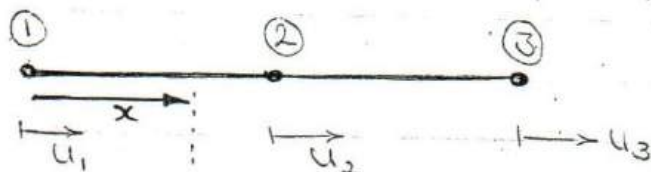
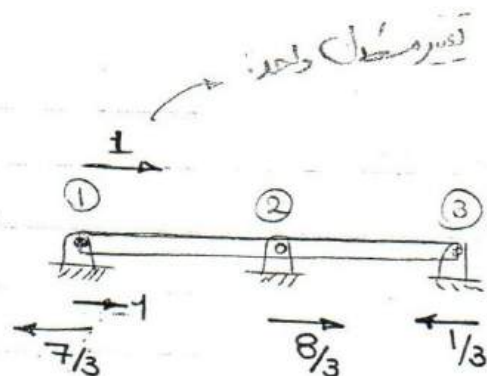
$$u(x) = \langle N_1 \ N_2 \ N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = N_2(x) \cdot u_2$$

$$= \left(\frac{4x}{L} - \frac{4x^2}{L^2} \right) \cdot \frac{PL^2}{8AE}$$

$$\Rightarrow AE \cdot \frac{d^2 u}{dx^2} + P(x) = 0$$

حل FEM با حل دقیق می است. چون حل دقیق تابع دیر شکل رادیه 2 به 2 ده

برای بدست آوردن سطر اول K → آسانتر مانتوسی
 مشق اول

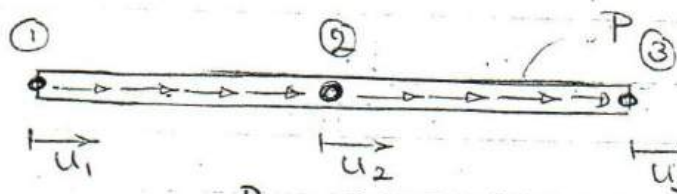


$$u(x) = a_0 + a_1 x + a_2 x^2$$

$$u(x) = N_1(x) u_1 + N_2(x) u_2 + N_3(x) u_3$$

$$K \cdot \underline{U} = \underline{F}$$

$$\underline{F} = \underline{F}_c + \underline{F}_q + \underline{F}_b + \underline{F}_e + \underline{F}_s$$

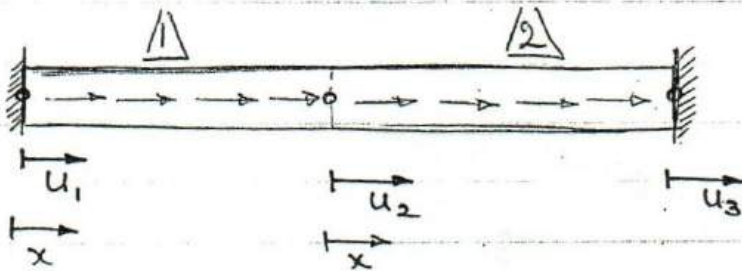


$$\underline{F}_q = \int_{(s)} N^T(\underline{q}) ds$$

$$\underline{F}_q = \int_0^L \begin{Bmatrix} 1 - \frac{3x}{L} + \frac{2x^2}{L^2} \\ \frac{4x}{L} - \frac{4x^2}{L^2} \\ -\frac{x}{L} + \frac{2x^2}{L^2} \end{Bmatrix} p \cdot dx = \frac{PL}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

مثال قبل را با دو تالمان گرهی حل کنیم.

حل: با دو تالمان گرهی.



$$\mathbf{P}_K^{\Delta} = \mathbf{P}_K^{\Delta} = \frac{AE}{L/2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{P}_q^{\Delta} = \mathbf{P}_q^{\Delta} = \frac{PL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

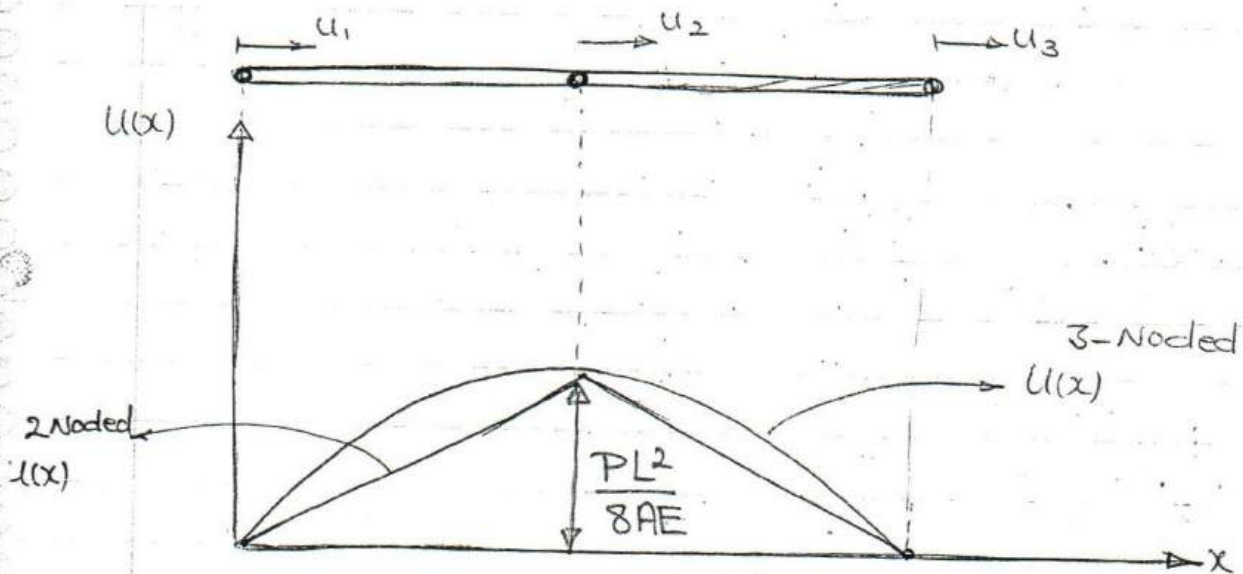
$$\frac{2AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \frac{PL}{4} \\ \frac{PL}{4} + \frac{PL}{4} \\ \frac{PL}{4} \end{Bmatrix}$$

$$\frac{4EA}{L} \cdot u_2 = \frac{PL}{2} \rightarrow u_2 = \frac{P \cdot L^2}{8AE} \rightarrow \text{تأثیری برابر شد}$$

$$u(x) = \left\langle 1 - \frac{x}{L} \quad \frac{x}{L} \right\rangle \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{x}{L} \left(\frac{PL^2}{8AE} \right)$$

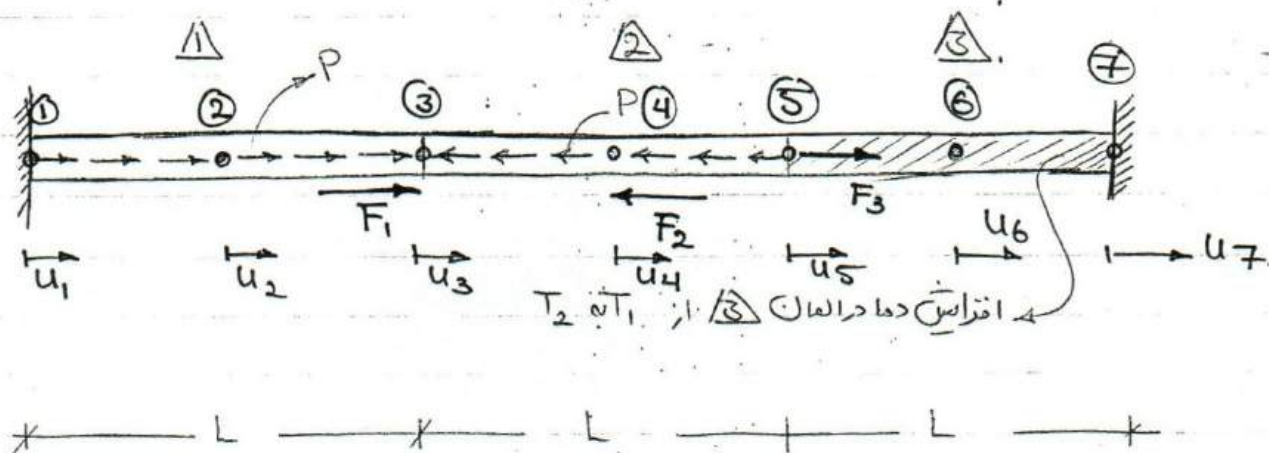
برای (حل فقط در وسط) و هم منطبق است.

$$\Delta \quad u(x) = \left\langle 1 - \frac{x}{L} \quad \frac{x}{L} \right\rangle \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \left(1 - \frac{x}{L}\right) \left(\frac{PL^2}{8AE}\right)$$



← این مسئله را با 3 امان دوری ، 4 امان دوری ، 5 امان دوری ، ... محدود حل کنید

مثال ۲) مطلوب است حل عضو محورى مثل زیر جهت بارهای اعمالی با استفاده از سه گان سه تری.



$$\rightarrow K_{7 \times 7} \cdot U_{7 \times 1} = F_{7 \times 1}$$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	7	-8	1	0	0	0	0
u_2	-8	16	-8	0	0	0	0
u_3	1	-8	7+7	-8	1	0	0
u_4	0	0	-8	16	-8	0	0
u_5	0	0	1	-8	7+7	-8	1
u_6	0	0	0	0	-8	16	-8
u_7	0	0	0	0	1	-8	7

$\frac{AE}{3L} \times$

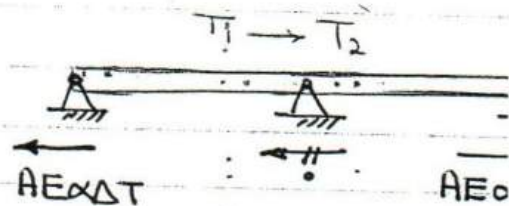
$$= \begin{matrix} \tilde{F}_c \\ \left\{ \begin{array}{c} 0 \\ 0 \\ F_1 \\ -F_2 \\ F_3 \\ 0 \\ 0 \end{array} \right\} \end{matrix} + \begin{matrix} \tilde{F}_q \\ \left\{ \begin{array}{c} PL/6 \\ 4PL/6 \\ PL/6 - PL/6 \\ -4PL/6 \\ -PL/6 \\ 0 \\ 0 \end{array} \right\} \end{matrix} + \begin{matrix} \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -AE\alpha\Delta T \\ AE\alpha\Delta T \end{array} \right\} \end{matrix}$$

$$\textcircled{3} \quad \tilde{F}_e = \int_V \tilde{B}^T : \tilde{D} : \tilde{\epsilon} \, dV =$$

$\xrightarrow{A \cdot dx}$
 $\xrightarrow{\epsilon_{ox} = \alpha(T_2 - T_1)}$
 $\xrightarrow{E}$

$$= \int_0^L \left\{ \begin{array}{c} -\frac{3}{L} + \frac{4x}{L^2} \\ \frac{4}{L} - \frac{8x}{L^2} \\ -\frac{1}{L} + \frac{4x}{L^2} \end{array} \right\} \cdot AE \cdot \alpha \cdot \Delta T \cdot dx =$$

$$= \left\{ \begin{array}{c} \textcircled{5} \\ \textcircled{6} \\ \textcircled{7} \end{array} \right\} \cdot AE \cdot \alpha \cdot \Delta T \cdot \left\{ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right\}$$



Static Condensation :

خذف استاتیکی

$$K \cdot U = F$$

unknown nodal displacements : U

restrained " " : معلومات

"Known"

$$U = \begin{Bmatrix} U_u \\ U_r \end{Bmatrix}$$

$$\begin{bmatrix} K_{uu} & K_{ur} \\ K_{ru} & K_{rr} \end{bmatrix} \begin{Bmatrix} \bar{U}_u \\ \bar{U}_r \end{Bmatrix} = \begin{Bmatrix} F_u \\ F_r \end{Bmatrix}$$

Known external loads F_u

reactions of support "unknown"

$$K_{uu} \cdot \bar{U}_u + K_{ur} \cdot \bar{U}_r = F_u$$

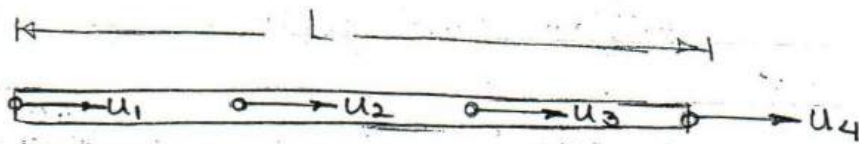
$$K_{ru} \cdot \bar{U}_u + K_{rr} \cdot \bar{U}_r = F_r$$

$$\bar{U}_u = K_{uu}^{-1} (F_u - K_{ur} \cdot \bar{U}_r)$$

$$F_r = K_{ru} \cdot K_{uu}^{-1} (F_u - K_{ur} \cdot \bar{U}_r) + K_{rr} \cdot \bar{U}_r$$

4. noded.

④ مرتبة



$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$u(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= \underline{P}(x) \cdot \underline{a}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{L}{3} & \frac{L^2}{9} & \frac{L^3}{27} \\ 1 & \frac{2L}{3} & \frac{4L^2}{9} & \frac{8L^3}{27} \\ 1 & L & L^2 & L^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$\leftarrow u(x)|_{x=0} = u_1$
 $\leftarrow u(x)|_{x=L/3} = u_2$
 $\leftarrow u(x)|_{x=2L/3} = u_3$
 $\leftarrow u(x)|_{x=L} = u_4$

$$\underline{C}_{4 \times 4} \cdot \underline{a} = \underline{\bar{u}} \longrightarrow \underline{a} = \underline{C}_{4 \times 4}^{-1} \cdot \underline{\bar{u}}$$

$$u(x) = \begin{bmatrix} \underline{P}(x) \cdot \underline{C}_{4 \times 4}^{-1} \cdot \underline{\bar{u}} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$\underline{N}(x) \leftarrow \underline{P}(x) \cdot \underline{C}_{4 \times 4}^{-1}$
 $= \underline{N}(x) \cdot \underline{\bar{u}}$

تغییر مکان (u) $\rightarrow$ نیروی فشار F

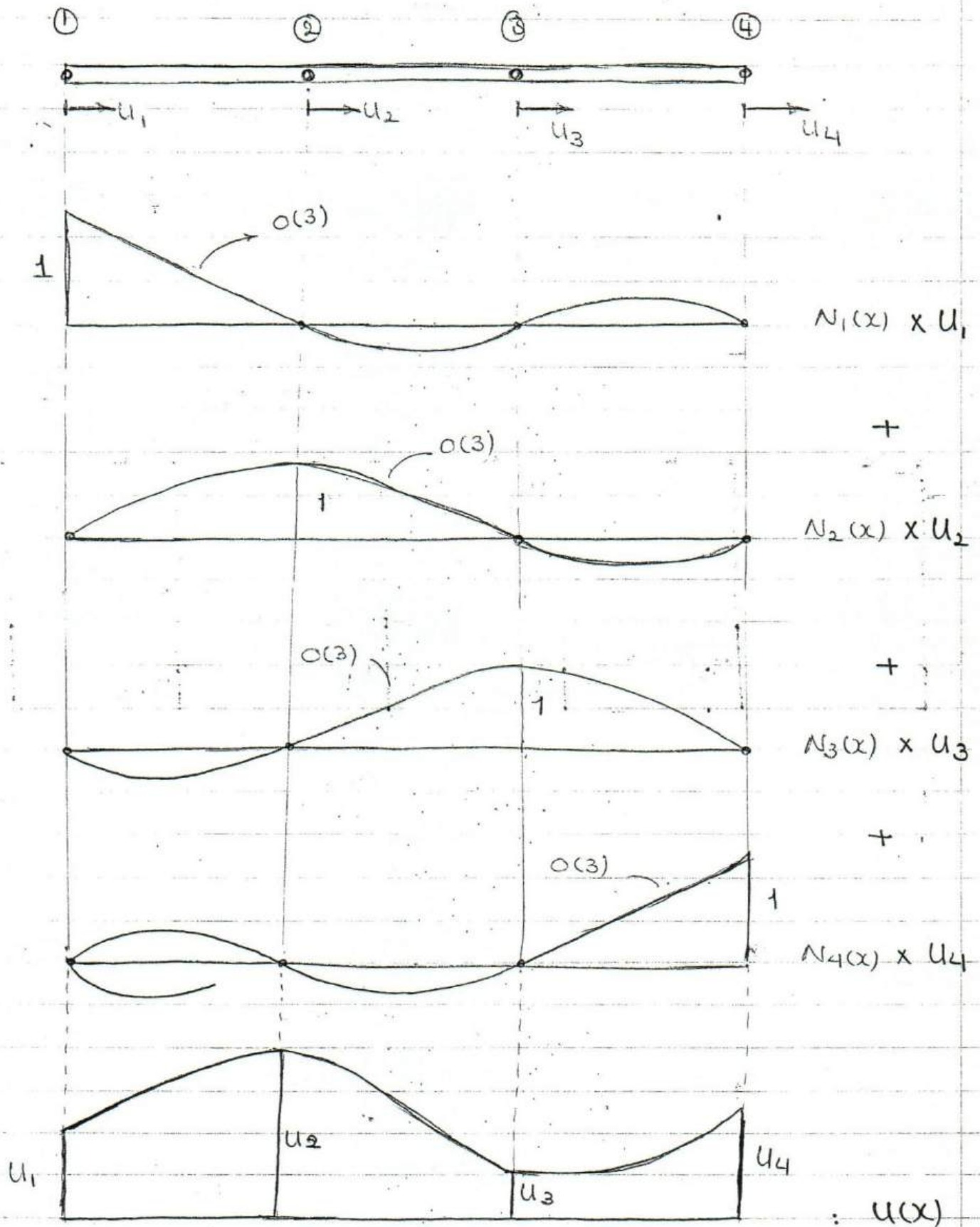
تغییر مکان U_u $\rightarrow$ معلوم $\Rightarrow$ گره‌های غیر پهنه‌های

معلوم U_p $\rightarrow$ معلوم $\Rightarrow$ پهنه‌های گره‌ها

معلوم U_p $\rightarrow$ معلوم $\Rightarrow$ نشست پهنه‌های

معلوم U_p $\rightarrow$ معلوم $\Rightarrow$ گره‌های غیر پهنه‌های
 که تغییر مکان آن
 گره الزاماً ثابت و معلوم
 است.

No.



$$u(x) = \sum_{i=1}^4 N_i u_i = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underline{\underline{N}}(x) \underline{\underline{u}}$$

Lagrange method.

لاگرانژ:

توانج
شکل لادانژ برای تعیین ~~شکل~~ ~~شکل~~ ~~شکل~~

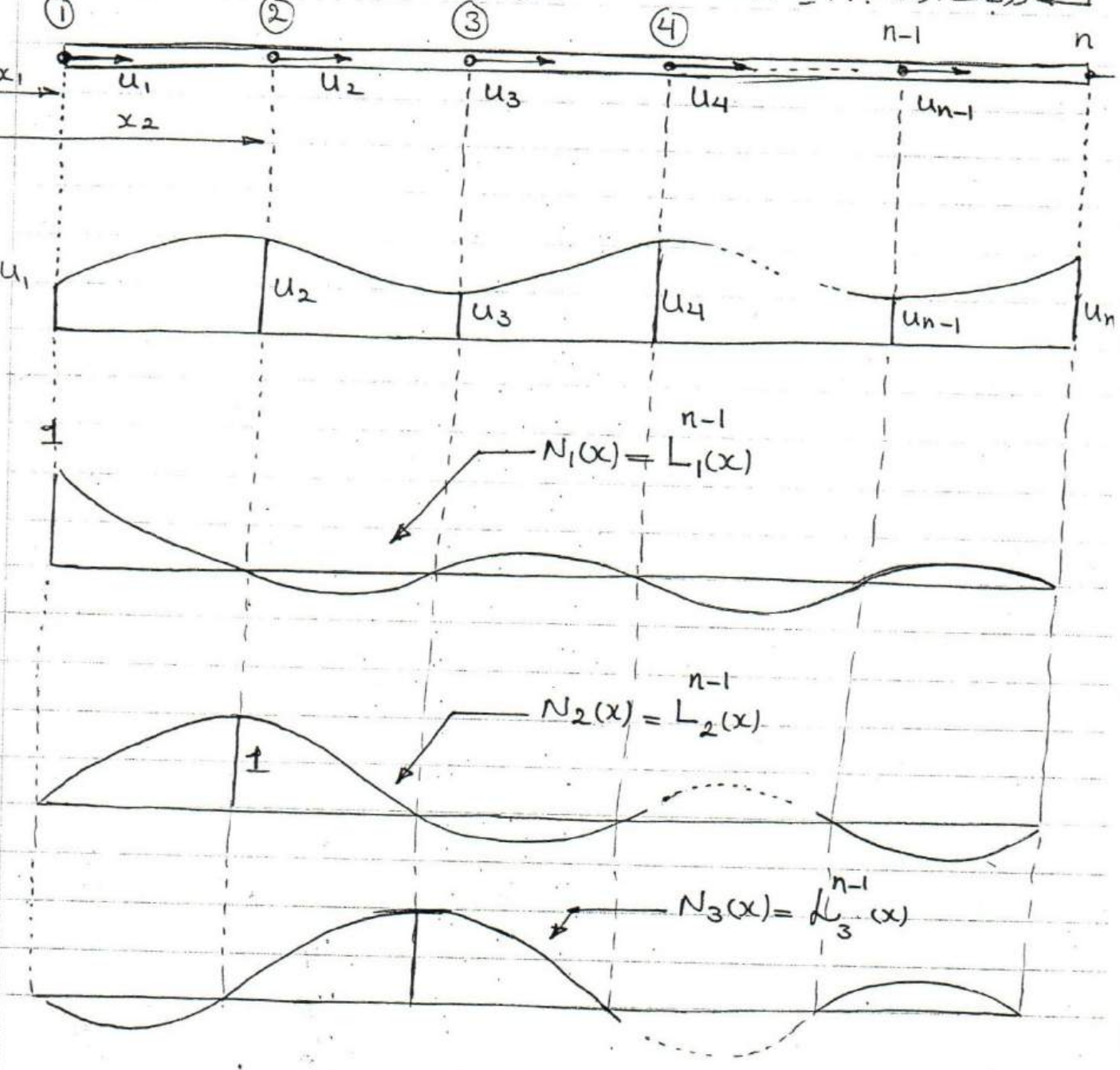
سیان جابه جایی $u(x)$

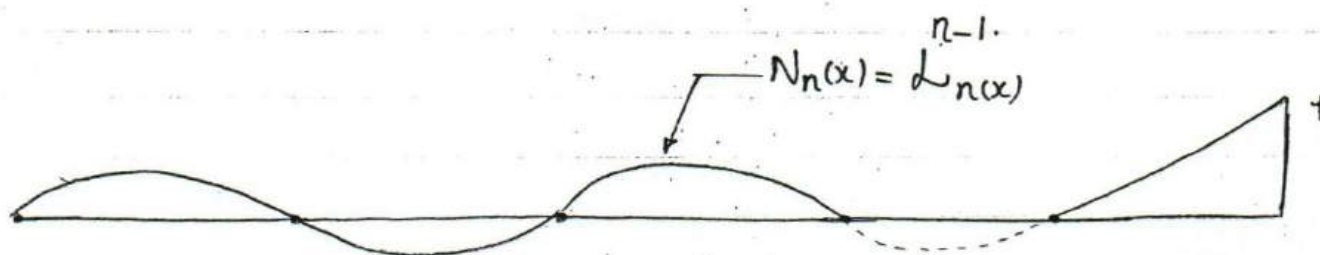
اس این روش هر تابع نه دارای مقدار مشخص در n نقطه باشد برای توان در اساس مجموعی از جابه جایی

بر مشخص در تابعی نه دارای مقدار واحد در نقطه و مقدار مشخص و صفر در سایر نقاط باشد

تعریف تابع شکل است

در جابه آزاری جابه جایی در برده ها



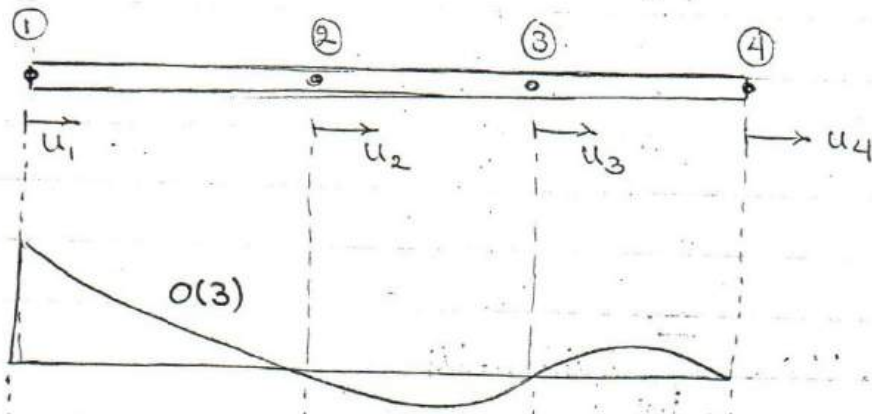


$$\rightarrow u(x) = \sum_{i=1}^n L_i(x) \cdot u_i = \sum_{i=1}^n N_i(x) \cdot u_i$$

$$\Rightarrow L_i(x) = \frac{(x-x_1)(x-x_2)(x-x_3) \dots (x-x_{i-1})(x-x_{i+1}) \dots (x-x_n)}{(x_i-x_1)(x_i-x_2) \dots (x_i-x_{i-1})(x_i-x_{i+1}) \dots (x_i-x_n)}$$

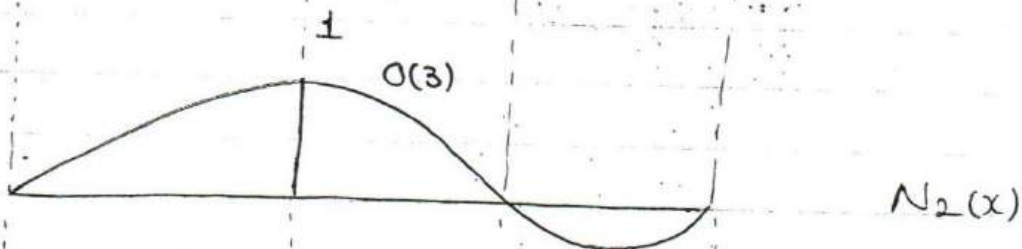
$$\rightarrow N_i(x) = L_i(x) = \frac{\prod_{j=1, j \neq i}^n (x-x_j)}{\prod_{j=1, j \neq i}^n (x_i-x_j)} \quad i \neq j$$

سوال مثال امان 4 گرهی :



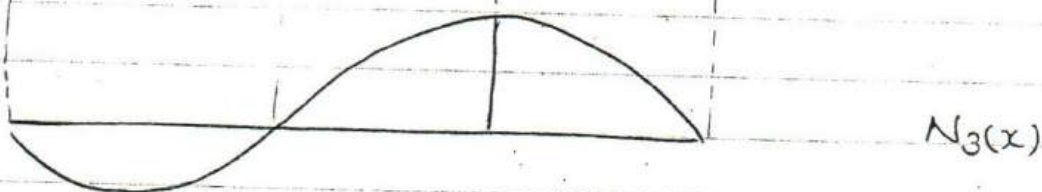
$$N_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$= \frac{(x-L/3)(x-2L/3)(x-L)}{(0-L/3)(0-2L/3)(0-L)}$$

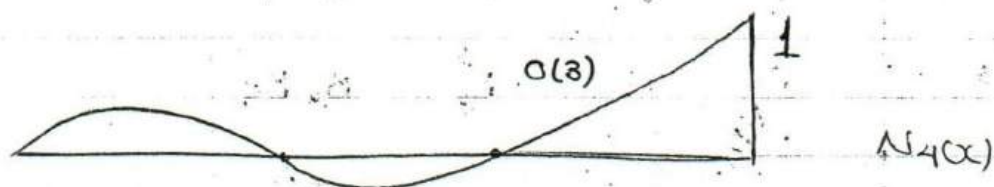


$$N_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$= \frac{(L/3-0)(L/3-2L/3)(L/3-L)}{(L/3-0)(L/3-2L/3)(L/3-L)}$$



$$N_3(x) = \frac{(x-0)(x-L/3)(x-L)}{(\frac{2L}{3}-0)(\frac{2L}{3}-\frac{L}{3})(\frac{2L}{3}-L)}$$



$$N_4(x) = \frac{(x-0)(x-L/3)(x-2L/3)}{(L-0)(L-L/3)(L-2L/3)}$$

$$\rightarrow u(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underline{N}(x) \cdot \underline{\bar{u}}$$

$$\underline{\epsilon} = \frac{du}{dx} = \langle \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \frac{dN_3}{dx} \quad \frac{dN_4}{dx} \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underline{B}(x) \underline{\bar{u}}$$

$$\underline{K}_{4 \times 4} = \int \underline{B}^T \underline{D} \underline{B} \, dV = \int \begin{Bmatrix} B_1(x) \\ B_2(x) \\ B_3(x) \\ B_4(x) \end{Bmatrix} \underline{E} \langle B_1(x) \quad B_2(x) \quad B_3(x) \quad B_4(x) \rangle \, dV$$

$$\underline{K} = \int_0^L B^T D B dV = \int_0^L AE \begin{bmatrix} B_1^2 & B_1 B_2 & B_1 B_3 & B_1 B_4 \\ & B_2^2 & B_2 B_3 & B_2 B_4 \\ & & B_3^2 & B_3 B_4 \\ & & & B_4^2 \end{bmatrix} dx$$

$\sigma_{1/3}$

آپرس سفق یک المان n برقی:

$$u(x) = \langle N_1(x) \quad N_2(x) \quad \dots \quad N_n(x) \rangle \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}$$

$$E(x) = \frac{du}{dx} = \langle \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \dots \quad \frac{dN_n}{dx} \rangle \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}$$

$B_1(x) \quad B_2(x) \quad \dots \quad B_n(x)$

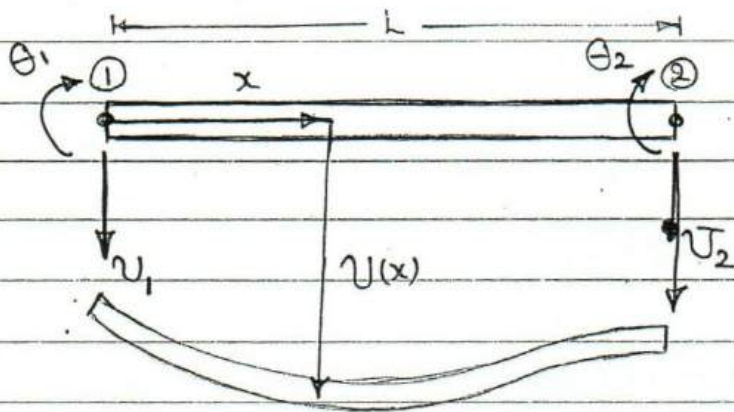
$$\underline{K}_{n \times n} = \int_0^L B^T D B dV = \int_0^L AE \begin{bmatrix} B_1^2 & B_1 B_2 & \dots & B_1 B_n \\ B_2 B_1 & B_2^2 & \dots & B_2 B_n \\ \vdots & \vdots & \ddots & \vdots \\ B_n B_1 & B_n B_2 & \dots & B_n^2 \end{bmatrix} dx$$

العناصر:

$$\rightarrow K \cdot \bar{u} = F \rightarrow F = F_c + F_q + F_{j_0} + F_e.$$

Beam. element

$$\rightarrow \begin{cases} y = \bar{N} \cdot \bar{u} \\ \epsilon = \bar{B} \cdot \bar{u} \end{cases}$$

العناصر (مستقيمة) بادولره و ۲ درجه آزادی (u, θ) در هر نره

$$\bar{u} = \bar{u} = \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

Rayleigh Ritz Function (1)

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= \langle 1 \ x \ x^2 \ x^3 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \bar{p}(x) \cdot \bar{a}$$

$$u(x) = \bar{p}(x) \cdot \bar{a}$$

Boundary condition. (2)

$$\theta = \frac{du(x)}{dx} = \langle 0 \ 1 \ 2x \ 3x^2 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$v(x) \Big|_{x=0} = v_1$$

$$\theta(x) \Big|_{x=0} = \theta_1$$

$$v(x) \Big|_{x=L} = v_2$$

$$\theta(x) \Big|_{x=L} = \theta_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

 $C_{4 \times 4}$

$$C_{4 \times 4} \cdot \underline{a} = \underline{v} \rightarrow \underline{v} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\underline{a} = C_{4 \times 4}^{-1} \underline{v}$$

$$v(x) = \underline{p}(x) \cdot \underline{a} = \underbrace{\underline{p}(x) \cdot C_{4 \times 4}^{-1}}_{\underline{N}(x)} \underline{v}$$

$$\underline{N}(x) = \langle N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x) \rangle \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\Pi = U + V = \int_V (\underbrace{U_0}_{\frac{1}{2} \sigma_{ij} \epsilon_{ij}}) dV - W_{ext} = \int_V U_0 dV - \left[\int_V b_i u_i dV + \int_S t_i u_i dS \right]$$

$$\delta \Pi = 0 \quad \rightarrow \quad u_i = \dots \quad \text{سلك الجاذبي معين}$$

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_V b_i \delta u_i dV - \int_S t_i \delta u_i dS = 0$$

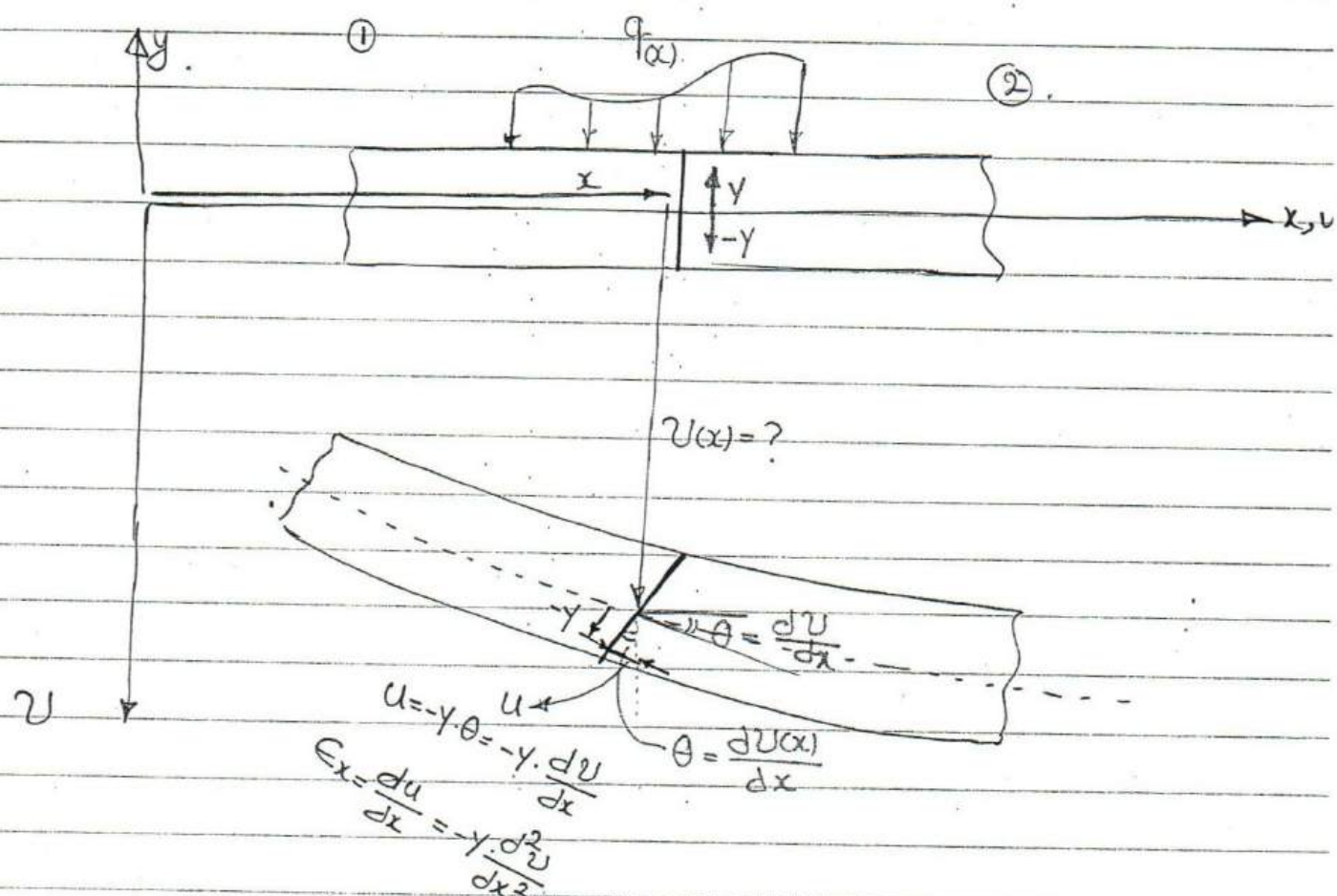
$$\rightarrow \text{نأخذ } \frac{1}{2} \rightarrow \frac{1}{2} \sigma_x \epsilon_x = \frac{1}{2} E \epsilon_x^2$$

$$\delta \Pi = E \epsilon_x \delta \epsilon_x = \sigma_x \epsilon_x$$

$$\text{نأخذ } \frac{1}{2} \sigma_x \epsilon_x = \frac{1}{2} E \epsilon_x^2 \rightarrow \delta u_x = E \epsilon_x \delta \epsilon_x = \sigma_x \epsilon_x$$

مطلوب است برای عضو خمشی شکل زیر که تحت بار عرضی $q(x)$ قرار گرفته. مطلوب است تعیین معادله جابجایی $v(x)$

استفاده از قضیه انرژی پتانسیل می‌نماید. «بدون نظر به شرح اثر بار» و ضمناً از افتادن محوری صرف نظر شود.



$$V = W_{ext} = - \int q(x) v(x) dx$$

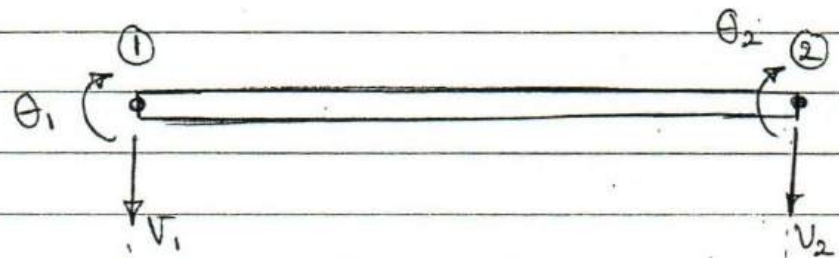
$$U = \int \frac{1}{2} \epsilon_{ij} \epsilon_{ij} dV = \int \frac{1}{2} E \epsilon_x^2 dV = \int \int \int \frac{1}{2} E \left(\frac{du}{dx} \right)^2 dA dx$$

$$\epsilon_x = \frac{du}{dx}$$

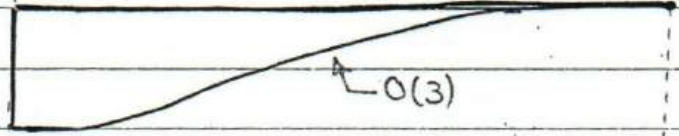
$$\epsilon_x = \frac{-y}{\rho} = -y \cdot \kappa = -y \frac{d^2v(x)}{dx^2}$$

دیت شود که N_1 و N_3 از یک وزن و N_2 و N_4 از یک وزن هستند.

21
A



$V_1 = 1$
 $\theta_1 = 0$

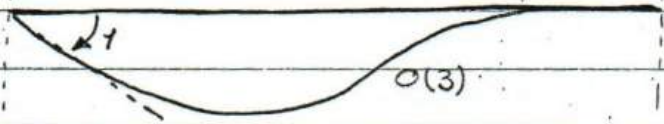


$V_2 = 0$
 $\theta_2 = 0$

$$N_1(x) = 1 + \frac{2x^3}{L^3} - \frac{3x^2}{L}$$

$N_1(x) * V_1$

$V_1 = 0$
 $\theta_1 = 1$

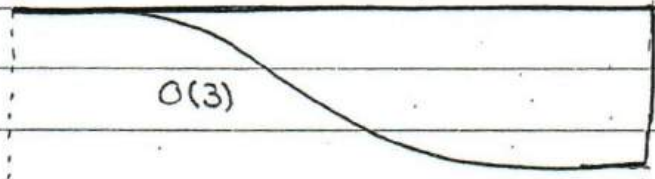


$V_2 = 0$
 $\theta_2 = 0$

$$N_2(x) = x \left(\frac{x}{L} - 1 \right)^2$$

$N_2(x) * \theta_1$

$V_1 = 0$
 $\theta_1 = 0$

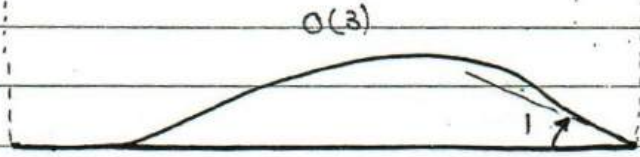


$V_2 = 1$
 $\theta_2 = 0$

$$N_3(x) = -\frac{2x^3}{L^3} + \frac{3x^2}{L^2}$$

$N_3(x) * V_2$

$V_1 = 0$
 $\theta_1 = 0$



$V_2 = 0$
 $\theta_2 = 1$

$$N_4(x) = x \left(\frac{x^2}{L^2} - \frac{x}{L} \right)$$

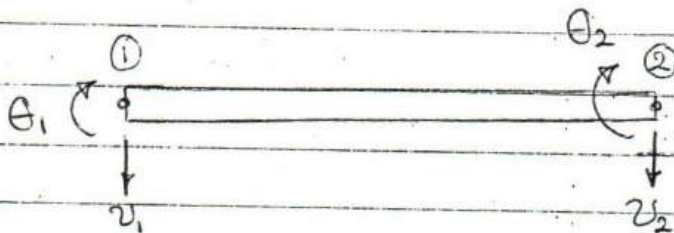
$N_4(x) * \theta_2$

$$E_x = \frac{B}{2} \cdot \bar{U} \rightarrow \begin{Bmatrix} U_1 \\ \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix}$$

$$E_x = \frac{du}{dx} = -\gamma \cdot \frac{d^2 u}{dx^2}$$

$$= -\gamma \left\langle \frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right\rangle \begin{Bmatrix} U_1 \\ \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix}$$

لغات تیر و گرهی « 4 درجه آزادی »



$$V(x) = P(x) \cdot C_{4 \times 4}^{-1} \cdot \bar{U} \rightarrow V(x) = \langle N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x) \rangle \begin{Bmatrix} U_1 \\ \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix}$$

$$\rightarrow N_1(x) = 1 + \frac{2x^3}{L^3} - \frac{3x^2}{L^2}$$

$$N_2(x) = x \left(\frac{x}{L} - 1 \right)^2$$

$$N_3(x) = -2 \frac{x^3}{L^3} + \frac{3x^2}{L^2}$$

$$N_4(x) = x \left(\frac{x^2}{L^2} - \frac{x}{L} \right)$$

$$\rightarrow \sum_{i=1,3} N_i(x) = 1$$

$$\epsilon_x = \frac{du}{dx} = \frac{d}{dx} \left(-y \cdot \frac{dV}{dx} \right) = -y \cdot \frac{d^2 V(x)}{dx^2}$$

$$= -y \underbrace{\left\langle \frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right\rangle}_{\tilde{B}(x)} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

$$\rightarrow \tilde{K} = \int_V \tilde{B}^T \cdot D \cdot \tilde{B} \, dV = \int_V \underbrace{\begin{pmatrix} -y \end{pmatrix}}_{\tilde{B}} \begin{Bmatrix} \frac{d^2 N_1}{dx^2} \\ \frac{d^2 N_2}{dx^2} \\ \frac{d^2 N_3}{dx^2} \\ \frac{d^2 N_4}{dx^2} \end{Bmatrix} \cdot E \cdot \begin{pmatrix} -y \end{pmatrix}^T \left\langle \frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \right\rangle$$

$$\frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \rangle \cdot |dA| \, dx$$

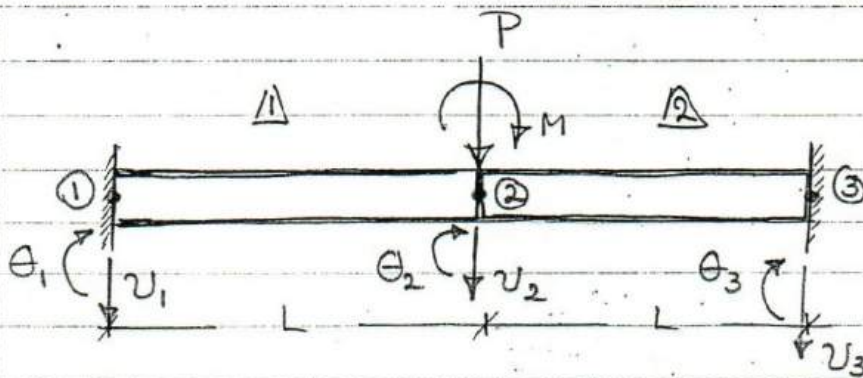
$$P_K = \int_0^L EI \left[\begin{array}{cccc} \left(\frac{d^2 N_1}{dx^2} \right)^2 & \frac{d^2 N_1}{dx^2} \frac{d^2 N_2}{dx^2} & \frac{d^2 N_1}{dx^2} \frac{d^2 N_3}{dx^2} & \frac{d^2 N_1}{dx^2} \frac{d^2 N_4}{dx^2} \\ \left(\frac{d^2 N_2}{dx^2} \right)^2 & \frac{d^2 N_2}{dx^2} \frac{d^2 N_3}{dx^2} & \frac{d^2 N_2}{dx^2} \frac{d^2 N_4}{dx^2} & \\ \left(\frac{d^2 N_3}{dx^2} \right)^2 & \frac{d^2 N_3}{dx^2} \frac{d^2 N_4}{dx^2} & & \\ \left(\frac{d^2 N_4}{dx^2} \right)^2 & & & \end{array} \right]$$

Sym.

$$P_{K_{4 \times 4}} = EI \left[\begin{array}{cc|cc} \textcircled{1} & & \textcircled{2} & \\ v_1 & \theta_1 & v_2 & \theta_2 \\ \hline \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ \hline -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \\ \hline & & & \end{array} \right] \begin{array}{l} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

①
②

مثال: برای تیر خمشی شکل زیر تحت بارهای P و m مطابق شکل مطلوب است میزان جابجایی و تنش در طول تیر. با استفاده از دو حالت تیر دگرهی.



$$K_{6 \times 6} \cdot U_{6 \times 1} = F_{6 \times 1}$$

v_1	θ_1	v_2	θ_2	v_3	θ_3	
$\frac{12}{L^3}$	$\frac{6}{L^2}$	$-\frac{12}{L^3}$	$\frac{6}{L^2}$	0	0	$v_1 = 0$
$\frac{6}{L^2}$	$\frac{4}{L}$	$-\frac{6}{L^2}$	$\frac{2}{L}$	0	0	$\theta_1 = 0$
$-\frac{12}{L^3}$	$-\frac{6}{L^2}$	$\frac{12}{L^3} + \frac{12}{L^3}$	$-\frac{6}{L^2} + \frac{6}{L^2}$	$-\frac{12}{L^3}$	$\frac{6}{L^2}$	v_2
$\frac{6}{L^2}$	$\frac{2}{L}$	$\frac{6}{L^2} - \frac{6}{L^2}$	$\frac{4}{L} + \frac{4}{L}$	$-\frac{6}{L^2}$	$\frac{2}{L}$	θ_2
0	0	$-\frac{12}{L^3}$	$-\frac{6}{L^2}$	$\frac{12}{L^3}$	$-\frac{6}{L^2}$	$v_3 = 0$
0	0	$\frac{6}{L^2}$	$\frac{2}{L}$	$-\frac{6}{L^2}$	$\frac{4}{L}$	$\theta_3 = 0$

$\left. \begin{matrix} P \\ M \end{matrix} \right\} F$

$$\begin{bmatrix} \frac{24EI}{L^3} & 0 \\ 0 & \frac{8EI}{L} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ M \end{Bmatrix}$$

$$\rightarrow v_2 = \frac{PL^3}{24EI} \quad \theta_2 = \frac{M.L}{8EI}$$

$$\Delta \quad U(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = N_3(x)v_2 + N_4\theta_2$$

$$\Delta \quad \rightarrow \epsilon_x = -y \left[\frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right]$$

$$\Delta \quad \rightarrow \sigma_x = \epsilon_x \cdot E = -y \left[\frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right] \cdot E$$

$$\Delta \quad U(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = N_1(x)v_2 + N_2(x) \cdot \theta_2$$

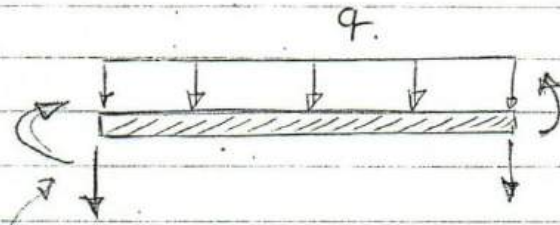
$$\Delta \quad \epsilon_x = -y \left[\frac{d^2 N_1(x)}{dx^2} v_2 + \frac{d^2 N_2}{dx^2} \theta_2 \right]$$

$$\Delta \quad \sigma(x) = E \cdot \epsilon_x$$

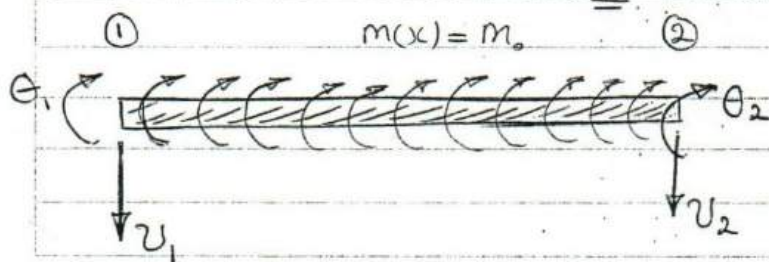
معادله‌های معادل گرهی ناشی از بار موزون باشد

$$\vec{F}_g = \int_S \vec{N}^T \vec{q} \, ds$$

$$= \int_0^L \begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{Bmatrix} \cdot q \, dx = \begin{Bmatrix} \frac{qL}{2} \\ \frac{qL^2}{12} \\ \frac{qL}{2} \\ -\frac{qL^2}{12} \end{Bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix}$$



ندر حقیقی بازنواخت باشد



$$m \cdot \delta \theta \rightarrow m \cdot \frac{d}{dx} (\delta v) \rightarrow m \cdot \left(\frac{dN}{dx} \delta \vec{v} \right)^T$$

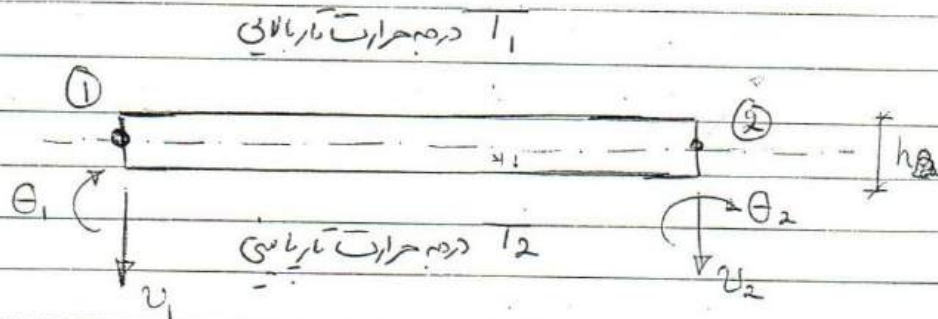
$$V = N \cdot \vec{v} \rightarrow \delta V = N \cdot \delta \vec{v}$$

$$\vec{F}_m = \int_0^L \left(\frac{dN}{dx} \right)^T m(x) \, dx$$

$$= \int_0^L \begin{Bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \\ \frac{dN_3}{dx} \\ \frac{dN_4}{dx} \end{Bmatrix} \cdot m \, dx = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix}$$

دارم تابل کدهی ناشی از تنش اولی E_0 (در اثر تغییر درجه حرارت و در ارتفاع سیر

در تغییر طول محوری ناشی از حرارت صاف می شود)



$$T(y) = \frac{T_1 + T_2}{2} - \frac{T_2 - T_1}{h} y$$

$$F_{E_0} = \int B^T D E_0 dV$$

$$= \int_L \int_A (-y) \begin{Bmatrix} \frac{d^2 N_1}{dx^2} \\ \frac{d^2 N_2}{dx^2} \\ \frac{d^2 N_3}{dx^2} \\ \frac{d^2 N_4}{dx^2} \end{Bmatrix} \cdot E \cdot \alpha \cdot T(y) \cdot dA \cdot dx = \begin{Bmatrix} \hat{V}_1 \\ \hat{M}_1 \\ \hat{V}_2 \\ \hat{M}_2 \end{Bmatrix}$$

??

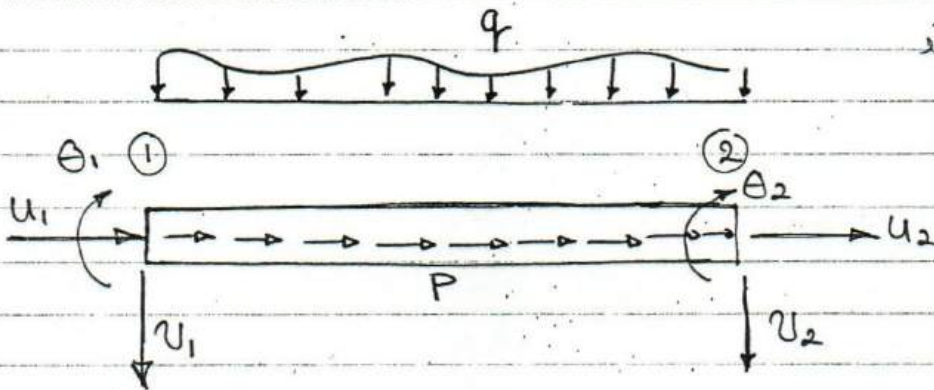
$$F_{E_0} = \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{Bmatrix} \frac{E \alpha I}{h} (T_2 - T_1)$$

Beam Column element

المان تیر-ستون

با فرض آن که نیروی محوری بر روی حقیقی ناشی از بار باشد و المان محوری و خمشی به سادگی

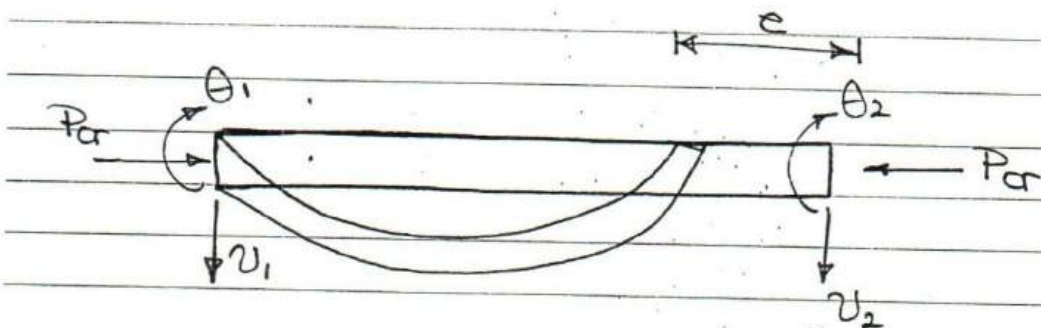
با یکدیگر در نظر گرفته می شوند



$$K = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \\ 0 & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

تقریب : مطلوب است تعیین ماتریس صفتی ناشی از لمانش در عضو خمشی شکل زیر « از المان 2 گره ای با

یا 2 درجه آزادی » « θ و v در هر گره استفاده کنید



$$K_T = K_{\text{beam}} + K_{\text{buckling}}$$

$$\int \delta \epsilon_x \cdot \sigma \, dv$$

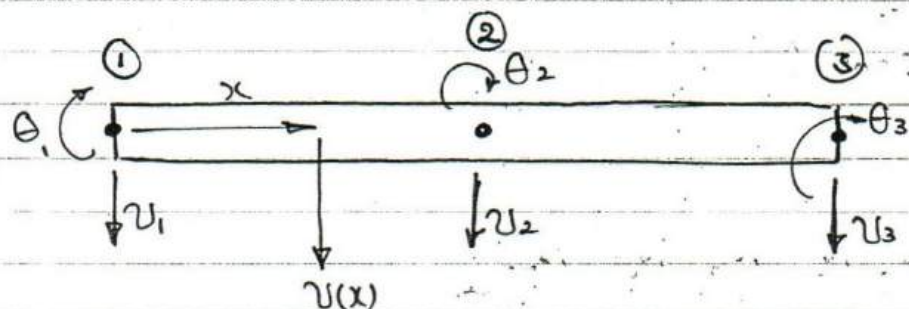
$$\Rightarrow P_{\alpha} * e(v_{\alpha})$$

$$\int B^T D B \, dv$$

$$U(x) = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

الماترئة بالائة ءةءة ءةءة :

ءالة اءة : بااضائه ءءوءة ءةءة ءءوءة الماة « EI, L »



$$v(x) = \langle 1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \tilde{P}(x) \cdot \tilde{a}$$

$$v(x) = \tilde{P}(x) \cdot \tilde{a}$$

$$\theta(x) = \langle 0 \quad 1 \quad 2x \quad 3x^2 \quad 4x^3 \quad 5x^4 \rangle \cdot \tilde{a} = \frac{d\tilde{P}(x)}{dx} \cdot \tilde{a}$$

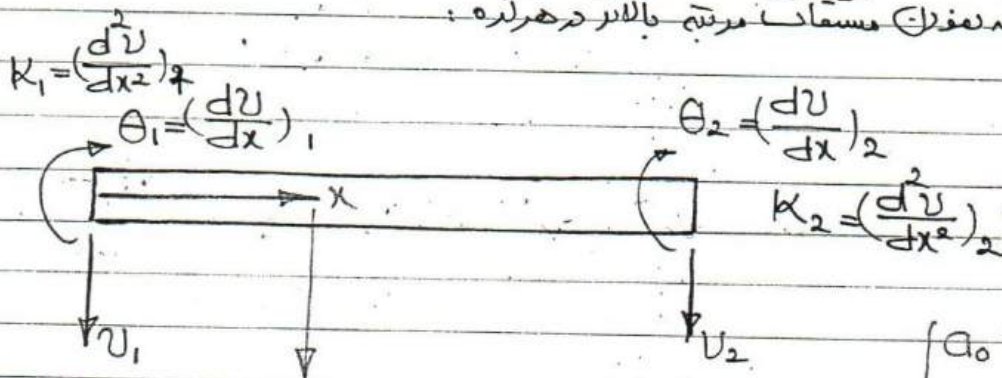
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} & \left(\frac{L}{2}\right)^3 & \left(\frac{L}{2}\right)^4 & \left(\frac{L}{2}\right)^5 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

$$\underline{C}_{6 \times 6} * \underline{a} = \underline{\bar{u}} \rightarrow \underline{a} = \underline{C}_{6 \times 6}^{-1} * \underline{\bar{u}}$$

$$u(x) = \underbrace{p(x) \cdot \underline{C}_{6 \times 6}^{-1}}_{N(x)} * \underline{\bar{u}}$$

$$= \langle N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \rangle_{(x)} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix}$$

حالت دوم: با اضافه نمودن مشتقات مرتبه بالاتر در هر تیر:



$$u(x) = \langle 1 \ x \ x^2 \ x^3 \ x^4 \ x^5 \rangle \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \underline{p(x)} \cdot \underline{a}$$

$$\rightarrow u(x) = \underline{p(x)} \cdot \underline{a}$$

$$\theta = \frac{du}{dx} = \langle 0 \ 1 \ 2x \ 3x^2 \ 4x^3 \ 5x^4 \rangle \cdot \underline{a}$$

$$K(x) = \frac{d^2u}{dx^2} = \langle 0 \ 0 \ 2 \ 6x \ 12x^2 \ 20x^3 \rangle \cdot \underline{a}$$

$\bar{C}_{6 \times 6}$

$$\begin{bmatrix}
 1 & & & & & \\
 & 1 & & & & \\
 & & 2 & & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
 \end{bmatrix}
 \begin{Bmatrix}
 a_0 \\
 a_1 \\
 a_2 \\
 a_3 \\
 a_4 \\
 a_5
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 v_1 \\
 \theta_1 \\
 K_1 \\
 v_2 \\
 \theta_2 \\
 K_2
 \end{Bmatrix}$$

$1 \quad L \quad L^2 \quad L^3 \quad L^4 \quad L^5$

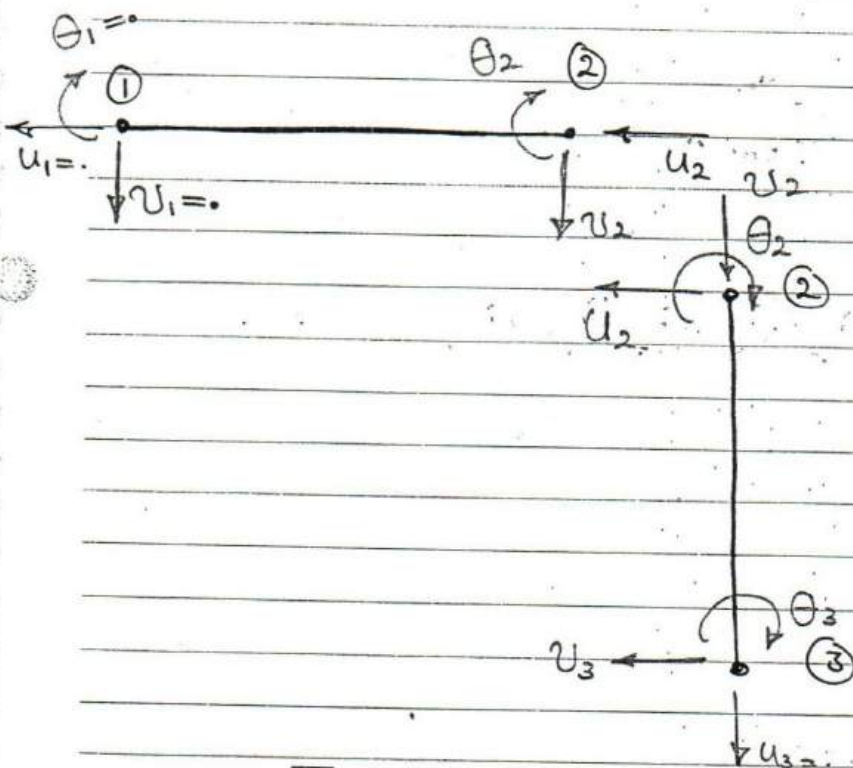
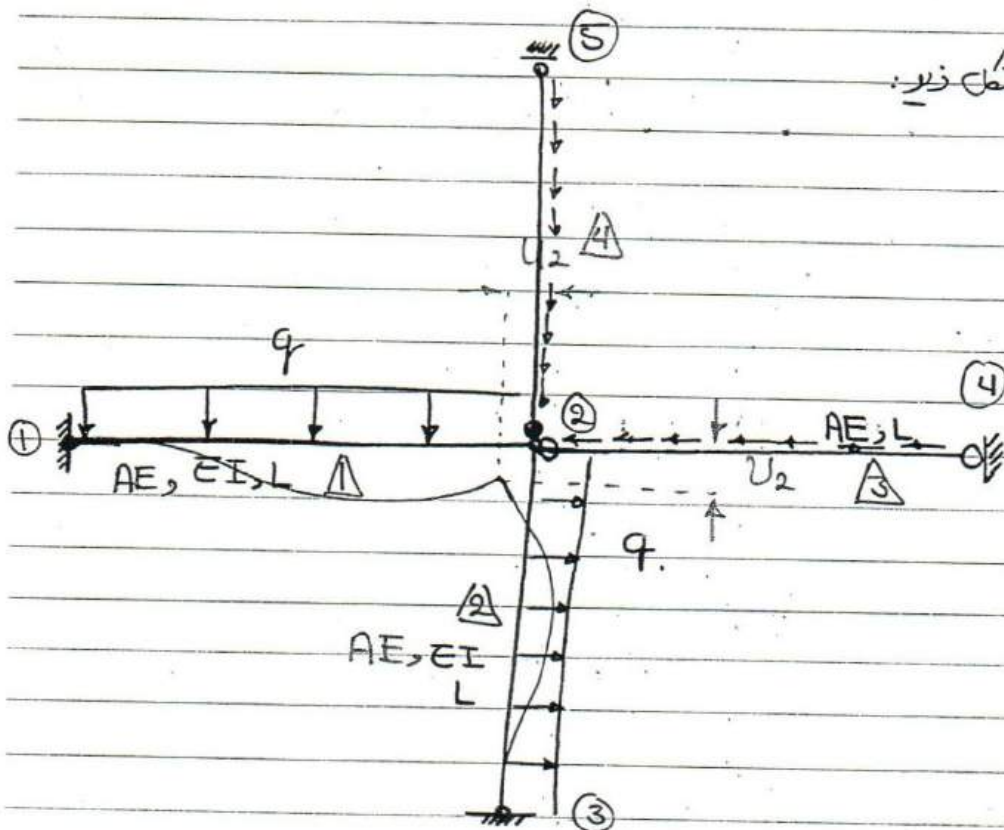
$$\bar{C}_{6 \times 6} * a = \hat{v} \rightarrow a = \bar{C}_{6 \times 6}^{-1} * \hat{v}$$

$$v(x) = \underset{\sim}{P}(x) \cdot \underset{\sim}{a} = \underset{\sim}{P}(x) \cdot \underset{\sim}{C}_{6 \times 6}^{-1} * \hat{v}$$

$N(x)$

$$= \langle \hat{N}_1 \hat{N}_2 \hat{N}_3 \hat{N}_4 \hat{N}_5 \hat{N}_6 \rangle \begin{Bmatrix} v_1 \\ \theta_1 \\ K_1 \\ v_2 \\ \theta_2 \\ K_2 \end{Bmatrix}$$

مطلوب است حل ماتریس سخت زیر:



$$\underset{\sim}{K}_{7 \times 7} \cdot \underset{\sim}{U}_{7 \times 1} = \underset{\sim}{F}_{7 \times 1} \longrightarrow \underset{\sim}{K}_{3 \times 3} \cdot \underset{\sim}{U}_{3 \times 1} = \underset{\sim}{F}_{3 \times 1}$$

$$\begin{bmatrix}
 \frac{12EI}{L^3} + \frac{AE}{L} + \frac{AE}{L} & \frac{6EI}{L^2} \\
 \frac{AE}{L} + \frac{12EI}{L^3} + \frac{AE}{L} & -\frac{6EI}{L^2} \\
 \frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{4EI}{L} + \frac{4EI}{L}
 \end{bmatrix}
 \begin{Bmatrix}
 u_2 \\
 v_2 \\
 \theta_2
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -\frac{qL}{2} + \frac{PL}{2} \\
 \frac{qL}{2} + \frac{PL}{2} \\
 \frac{(-q)L^2}{12} + \frac{-qL^2}{12}
 \end{Bmatrix}$$

Diagram of a beam of length \$L\$ with a uniformly distributed load \$q\$ acting downwards. The beam is fixed at both ends. The degrees of freedom are labeled as follows:

- Node 1 (Left): \$v_1\$ (vertical displacement), \$\theta_1\$ (rotation)
- Node 2 (Right): \$v_2\$ (vertical displacement), \$\theta_2\$ (rotation)

The stiffness matrix \$K\$ is given by:

$$K = EI \begin{bmatrix}
 \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\
 \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\
 -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\
 \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L}
 \end{bmatrix}$$

$$\rightarrow F_q = \begin{Bmatrix} \frac{qL}{2} \\ \frac{qL^2}{12} \\ \frac{qL}{2} \\ -\frac{qL^2}{12} \end{Bmatrix}$$

Diagram of a bar of length \$L\$ with axial displacements \$u_1\$ and \$u_2\$ at the ends.

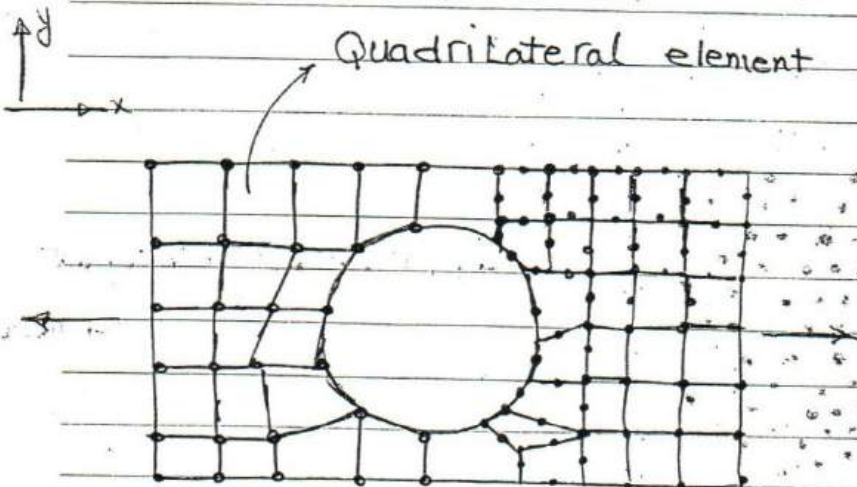
$$K = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Subject.....

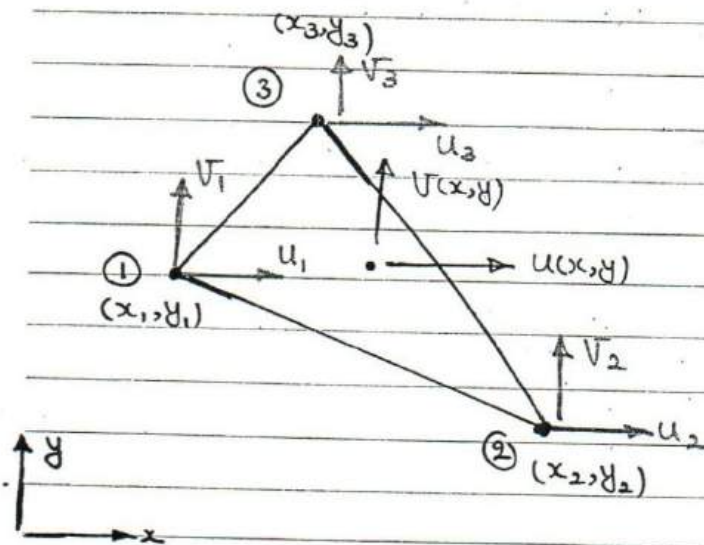
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المان 4 گوشه

Quadrilateral element



پرسش در المان مثلثی مثلثی تعریف می شود چرا؟



$$\bar{U}_2 = \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_2 \\ v_2 \\ \vdots \\ u_3 \\ v_3 \end{Bmatrix} \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}$$

تعیین جواب شرط برای این جواب درونی و کاربرد روش مستقیم

Step ① :

$$u(x, y) = a_0 + a_1 x + a_2 y$$

$$v(x, y) = b_0 + b_1 x + b_2 y$$

$$(1-a) u(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \tilde{P}(x, y) \cdot \tilde{a}$$

$$(1-b) v(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} b_0 \\ b_1 \\ b_2 \end{Bmatrix} = \tilde{P}(x, y) \cdot \tilde{b}$$

Step 2: Boundary Condition

$$u(x, y)|_{①} = u_1$$

$$u(x, y)|_{②} = u_2$$

$$u(x, y)|_{③} = u_3$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{\tilde{C}} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\tilde{C} \cdot \tilde{a} = \tilde{u}$$

$$\underline{a} = \underline{C}^{-1} \cdot \underline{\bar{U}} \quad (2-a)$$

$$\left. \begin{array}{l} (1-a) \\ (2-a) \end{array} \right\} \rightarrow u(x,y) = \underbrace{P(x,y)}_{N(x,y)} \cdot \underline{C}^{-1} \cdot \underline{\bar{U}}$$

$$N(x,y) = \langle N_1 \quad N_2 \quad N_3 \rangle$$

$$u(x,y) = \langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \sum_{i=1}^3 N_i(x,y) \cdot \bar{u}_i$$

(1-b)

$$V(x,y) \Big|_{\textcircled{1}} = U_1$$

$$V(x,y) \Big|_{\textcircled{2}} = U_2 \Rightarrow \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{\underline{C}} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}}_{\underline{\bar{U}}}$$

$$V(x,y) \Big|_{\textcircled{3}} = U_3$$

$$\underline{C} \cdot \underline{b} = \underline{\bar{U}} \rightarrow \underline{b} = \underline{C}^{-1} \cdot \underline{\bar{U}}$$

$$V(x,y) = \underbrace{P(x,y)}_{N(x,y)} \cdot \underline{C}^{-1} \cdot \underline{\bar{U}}$$

$$U(x, y) = \underbrace{P(x, y) \cdot C^{-1}}_{N(x)} \bar{U}$$

$$= \langle N_1 \quad N_2 \quad N_3 \rangle \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \sum_{i=1}^3 N_i(x, y) \cdot U_i$$

$$U(x, y) = N(x, y) \bar{U}$$

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_1 \\ \vdots \\ U_2 \\ U_2 \\ U_3 \\ U_3 \end{Bmatrix}$$

①
②
③
①
②
③

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \sum_{i=1}^3 \begin{bmatrix} N_i(x, y) & 0 \\ 0 & N_i(x, y) \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \end{Bmatrix}$$

$$= \sum_{i=1}^3 N_i(x, y) \bar{U}_i$$

$$C_{3 \times 3}^{-1} = \frac{1}{\det[C]} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

2 * A ←

$$\gamma_1 = \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

$$\gamma_2 = \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}$$

سفع المان ←

$$\det[C] = 2 * A = 2 * \text{سفع المان}$$

$$\gamma_3 = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$$

$$\alpha_1 = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$\alpha_2 = - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$\beta_1 = - \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix}$$

$$\beta_2 = \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix}$$

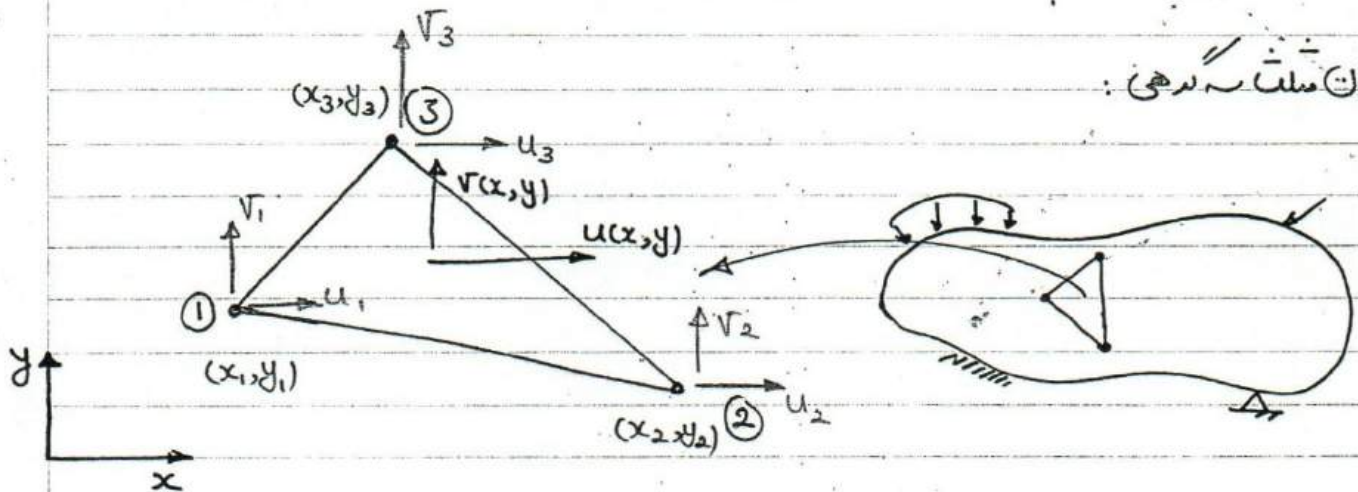
$$\beta_3 = - \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}$$

$$N(x, y) = \vec{P}(x, y) \cdot C^{-1}$$

$$= \langle 1 \ x \ y \rangle \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$$= \left\langle \frac{1}{2A} (\alpha_1 + \beta_1 x + \gamma_1 y) \quad \frac{1}{2A} (\alpha_2 + \beta_2 x + \gamma_2 y) \right\rangle$$

$$\frac{1}{2A} (\alpha_3 + \beta_3 x + \gamma_3 y) \rightarrow$$

 N_3


$$\bar{u} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

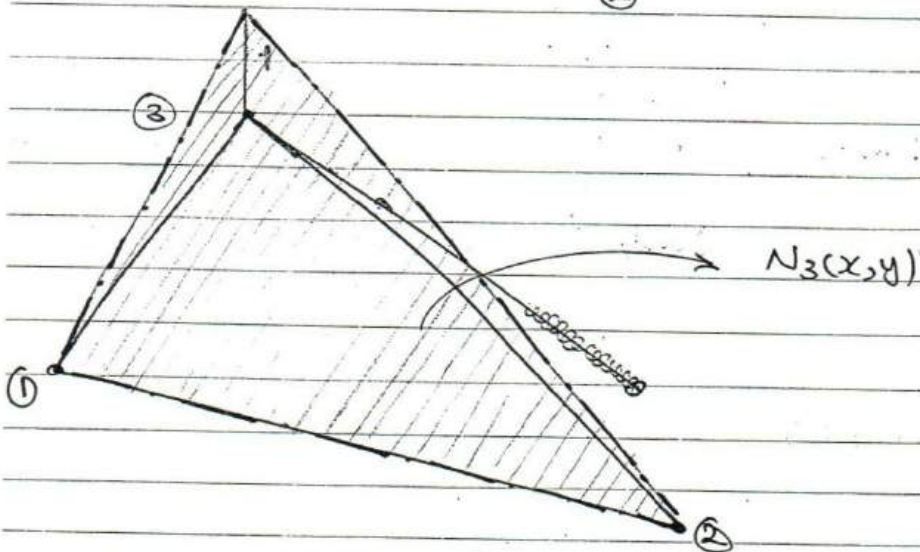
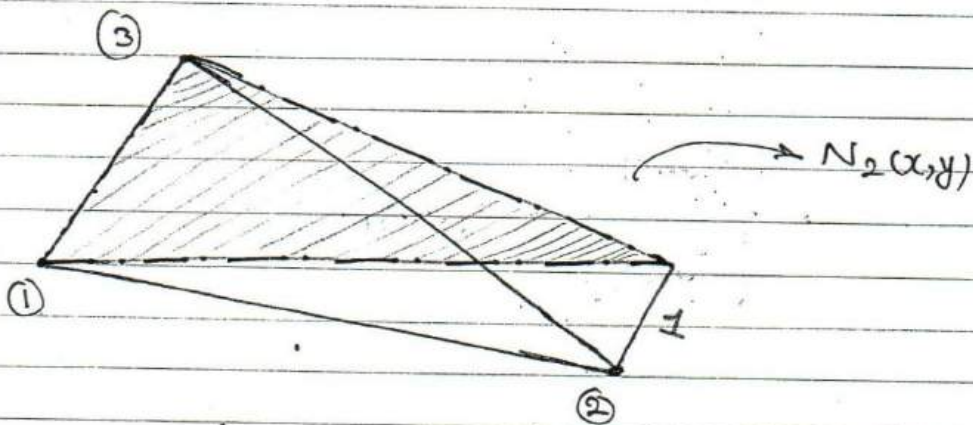
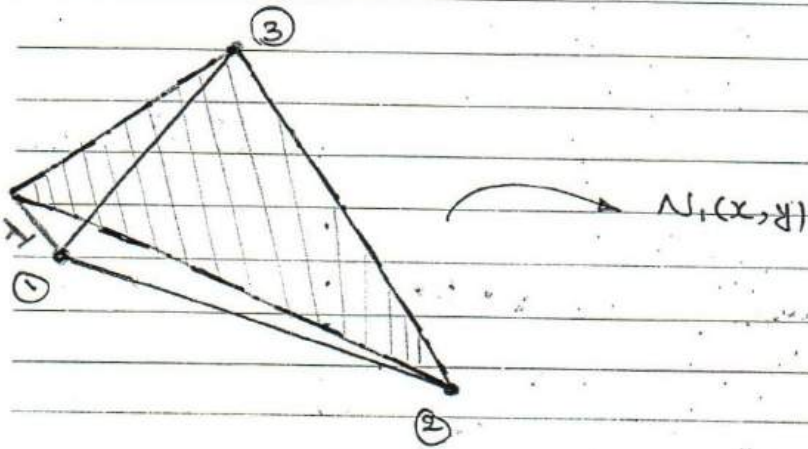
$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \underline{N}_{2 \times 6}(x,y) \cdot \bar{u}_{6 \times 1}$$

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \sum_{i=1}^3 \begin{bmatrix} N_i(x,y) & . \\ . & N_i(x,y) \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

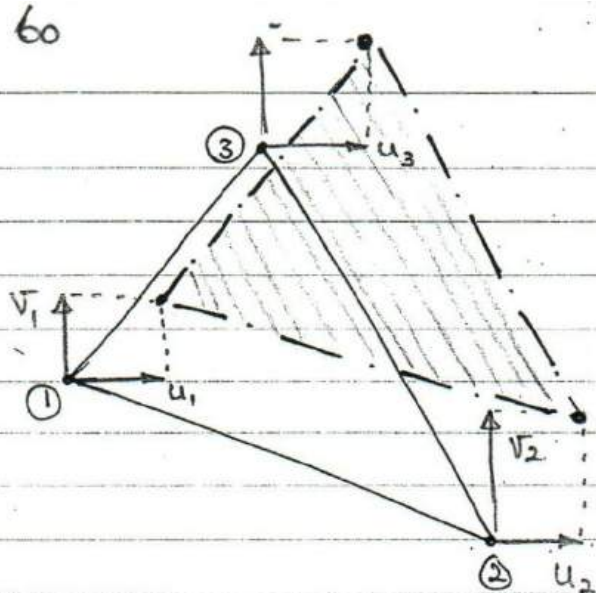
$$N_i(x,y) = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y) \quad i=1,2,3$$

$$u(x,y) = \sum_{i=1}^3 N_i \bar{u}_i = \langle N_1 \ N_2 \ N_3 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$v(x,y) = \sum_{i=1}^3 N_i \bar{v}_i = \langle N_1 \ N_2 \ N_3 \rangle \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$



$$\sum_{i=1}^3 N_i = 1$$



برای المان‌های مرتبه بالا مثلثی: المان مرتبه دوم - 6 درجه.

شکل ثابت

شکل‌های

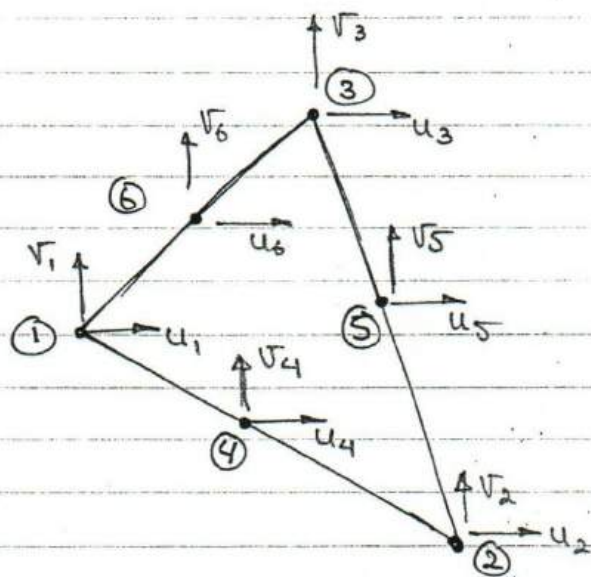
مرتبه دوم

$$u(x, y) = \bar{a}_0 + \bar{a}_1 x + \bar{a}_2 y + \bar{a}_3 x^2 + \bar{a}_4 xy + \bar{a}_5 y^2$$

$$v(x, y) = \bar{b}_0 + \bar{b}_1 x + \bar{b}_2 y + \bar{b}_3 x^2 + \bar{b}_4 xy + \bar{b}_5 y^2$$

$$\rightarrow P(x, y) = \langle 1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \rangle$$

$$P(x, y) = \langle 1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \rangle$$



$$\bar{U} = \left\{ \begin{array}{l} u_1 \\ v_1 \\ \vdots \\ u_2 \\ v_2 \\ \vdots \\ u_3 \\ v_3 \\ \vdots \\ u_4 \\ v_4 \\ \vdots \\ u_5 \\ v_5 \\ \vdots \\ u_6 \\ v_6 \end{array} \right\} \begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \\ \textcircled{5} \\ \textcircled{6} \end{array}$$

12x1

$$u(x, y) = P(x, y) \cdot \underline{a}$$

فریم محاسباتی بالایی دارد

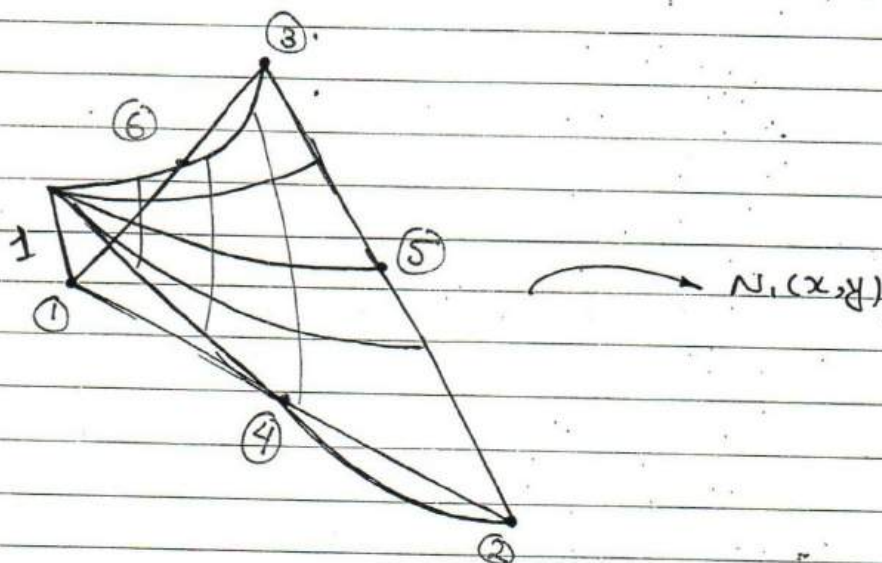
$$\underline{C}_{6 \times 6} \cdot \underline{a} = \underline{\bar{U}}_{6 \times 1} \rightarrow \underline{a} = \underline{C}_{6 \times 6}^{-1} \times \underline{\bar{U}}_{6 \times 1} \rightarrow \underline{\bar{U}} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{Bmatrix}$$

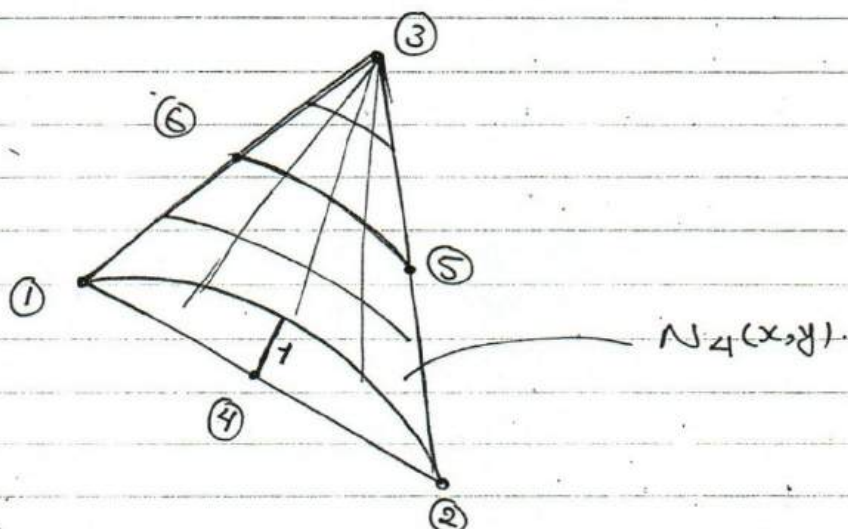
$$\rightarrow u(x, y) = P(x, y) \cdot \underline{C}_{6 \times 6}^{-1} \times \underline{\bar{U}}$$

$$\begin{cases} u(x, y) = \underline{N}(x, y) \cdot \underline{\bar{U}} = \langle N_1, N_2, \dots, N_6 \rangle \cdot \underline{\bar{U}} \\ v(x, y) = \underline{N}(x, y) \cdot \underline{\bar{V}} = \langle N_1, N_2, \dots, N_6 \rangle \cdot \underline{\bar{V}} \end{cases}$$

→ NODD : number of node

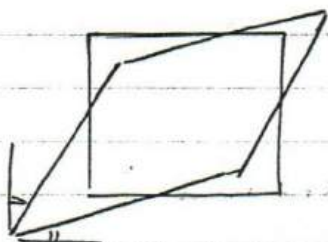
$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \sum_{i=1}^6 \begin{bmatrix} N_i(x, y) \cdot \\ \cdot \quad N_i(x, y) \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$





$$\underline{\epsilon} = \underline{B} \underline{\bar{U}} \longrightarrow \underline{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$\xrightarrow{\sum N_i u_i}$
 $\xrightarrow{\sum N_i v_i}$
 $\xrightarrow{\sum N_i u_i}$
 $\xrightarrow{\sum N_i v_i}$



$$= \sum_{i=1}^3 \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$\xrightarrow{\underline{\bar{U}}_i}$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$\xrightarrow{\underline{\bar{U}}_i}$

$$\underline{B}_i = \frac{1}{2A} \begin{bmatrix} B_1 & 0 & B_2 & 0 & B_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & B_1 & \gamma_2 & B_2 & \gamma_3 & B_3 \end{bmatrix}$$

$$\rightarrow \underline{B}_i = \frac{1}{2A} \sum_{i=1}^3 \begin{bmatrix} B_i & 0 \\ 0 & \gamma_i \\ \gamma_i & B_i \end{bmatrix}$$

$$\Rightarrow K = \int B^T \cdot D \cdot B \cdot dV$$

Constant Strain Element $\Rightarrow$ المان مشقی «ثابت»

C.S.E. element

در این المان به B_i و γ_i ها وابسته است و B_i ها و γ_i ها مقدار ثابتی دارند بنابراین در این المان مشقی «ثابت» است. چون B همواره ثابت است. یعنی در این المان، B ثابت است.

فرض می‌کنیم

$$t \cdot dA \Rightarrow dV = \underbrace{dx \cdot dy}_{dA} \cdot \underbrace{dz}_t$$

$$K = \int B^T \cdot D \cdot B \cdot dV$$

plane stress «Thickness = t»

$$\sigma_z = 0, \epsilon_z \neq 0$$

Plane strain $\rightarrow t = \infty$

$$\sigma_z \neq 0, \epsilon_z = 0$$

$$\sigma = D \cdot \epsilon$$

plane stress $\Rightarrow D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

plane strain $\Rightarrow D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$

$$K = \int_A \underbrace{B^T}_{6 \times 6} \underbrace{D}_{6 \times 3} \underbrace{B}_{3 \times 6} \underbrace{t \cdot dA}_{3 \times 6}$$

$$K_{ij} = \int_A \underset{2 \times 2}{\tilde{B}_i} \underset{2 \times 3}{D} \underset{3 \times 3}{\tilde{B}_j} t \, dA \quad i, j = 1, 2, 3$$

$$K = \underset{6 \times 6}{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}} \begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ \textcircled{1} & & & & & \\ & \textcircled{2} & & & & \\ & & \textcircled{3} & & & \end{matrix}$$

$\swarrow \quad \searrow$
 $2 \times 2 \quad \quad 2 \times 2$

$\Rightarrow$ هر کدام از این K ها یک ماتریس 2×2 هستند

محاسبه K با فرض شرایط Plane stress

$$\underset{2 \times 2}{K_{ij}} = \int_A \frac{1}{2A} \begin{bmatrix} \tilde{B}_i & 0 & \gamma_i \\ 0 & \gamma_i & \tilde{B}_i \end{bmatrix} \underset{2 \times 2}{\frac{E}{1-\nu^2}} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \tilde{B}_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \tilde{B}_j \end{bmatrix} \underset{2 \times 2}{\frac{1}{2A}}$$

$* t * dA$

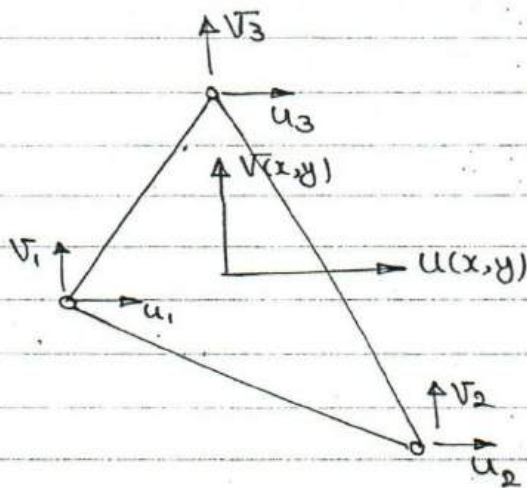
$$\gamma_i \beta_j = \beta_i \gamma_j \quad ??$$

$$\rightarrow \underset{2 \times 2}{\tilde{K}_{ij}} = \frac{E \cdot t}{4A(1-\nu^2)} \begin{bmatrix} \beta_i \beta_j + \frac{1-\nu}{2} \gamma_i \gamma_j & \nu \beta_i \gamma_j + \frac{1-\nu}{2} \gamma_i \beta_j \\ \nu \gamma_i \beta_j + \frac{1-\nu}{2} \beta_i \gamma_j & \gamma_i \gamma_j + \frac{1-\nu}{2} \beta_i \beta_j \end{bmatrix}$$

$$\text{مفرد اول} \Rightarrow i=1 \Rightarrow j=1,2,3 \Rightarrow K_{11} \quad K_{12} \quad K_{13} \quad i,j=1,2,3$$

$$\text{مفرد دوم} \Rightarrow i=2 \Rightarrow j=1,2,3 \Rightarrow K_{21} \quad K_{22} \quad K_{23}$$

$$\text{مفرد سوم} \Rightarrow i=3 \Rightarrow j=1,2,3 \Rightarrow K_{31} \quad K_{32} \quad K_{33}$$



$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{(x,y)} = \sum_{i=1}^{n \text{ nodes}} \begin{bmatrix} N_i & \cdot \\ \cdot & N_i \end{bmatrix}_{(x,y)} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$N_i(x, y) = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y) \quad i = 1, 2, 3$$

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \sum_{i=1}^{n \text{ nodes}} \begin{bmatrix} \frac{\partial N_i}{\partial x} & \cdot \\ \cdot & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \sum_{i=1}^3 \frac{1}{2A} \begin{bmatrix} \beta_i & \cdot \\ \cdot & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$K_{ij} = \int_{2 \times 2} \underline{B}_i^T \underline{D}_{3 \times 3} \underline{B}_j \cdot t \cdot dA \quad i, j = 1, 2, \dots, \text{NODE} \quad (2)$$

برای المان مرتبه دوم گاهی:

$$N_i(x, y) \quad (i = 1, 2, \dots, 6) \rightarrow \underline{N}_{2 \times 12} \rightarrow ?$$

$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$	$(i = 1, 2, \dots, 6)$	3×2 B_i همواره است
--	------------------------	----------------------------------

B_i همیشه 2×2 است و به تعداد Node ها

المان مثلثی نسبتی ندارد

B_i کل المان مثلثی گاهی 3×12 است

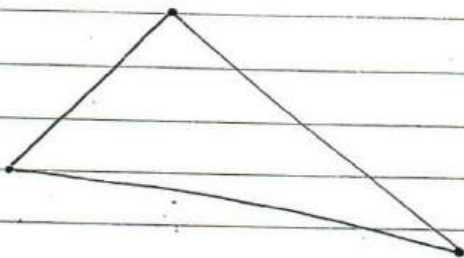
$$K_{ij} = \int_A \underline{B}_i^T \underline{D} \underline{B}_j \cdot t \cdot dA \quad i = 1, 2, \dots, 6$$

$$K_{12 \times 12} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & & & & K_{26} \\ \vdots & & & & & \\ K_{61} & K_{62} & & & & K_{66} \end{bmatrix}$$

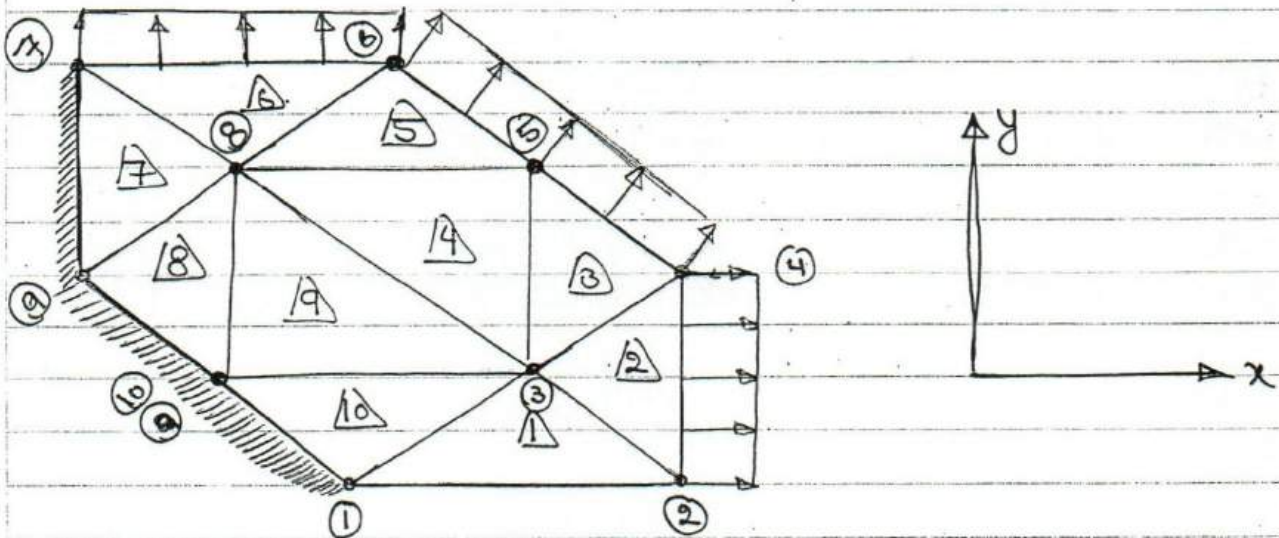
2×2 2×2

هر کدام از K_{ij} حالتی 2×2 است

همواره 2×2 است



مطلوب است نحوه تشکیل ماتریس سختی نشان زیر:



① Connectivity

② Coordinate

برای هر فایل شبکه المان محدود 2 مشخصه لازم است

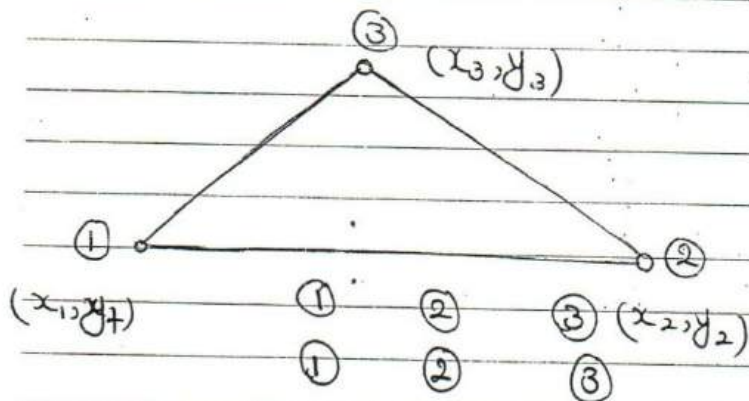
④ Connectivity.

element $\triangle$	node ①	node ②	node ③
① $\triangle$	①	②	③
② $\triangle$	②	④	③
③ $\triangle$	③	④	⑤
⋮			
⑩ $\triangle$	①	③	⑩

② Coordinate

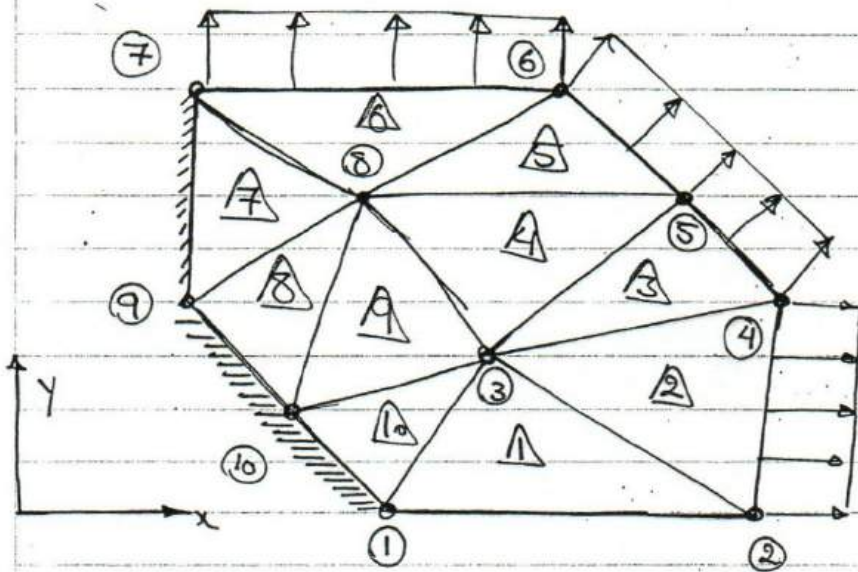
node	x-node	y-node
①	x_1	y_1
②	x_2	y_2
③	x_3	y_3
⋮		
⑩	x_{10}	y_{10}

$$\underset{\substack{2 \times 2 \\ 2 \times 1}}{\tilde{K}} \cdot \underset{2 \times 1}{\tilde{U}} = \underset{2 \times 1}{\tilde{F}}$$



$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{matrix} \text{①} & \text{②} & \text{③} \\ \text{①} & \text{②} & \text{③} \end{matrix}$$

مطلوب است نحوه تشکیل ماتریس سفتی مثل زیر



① Connectivity

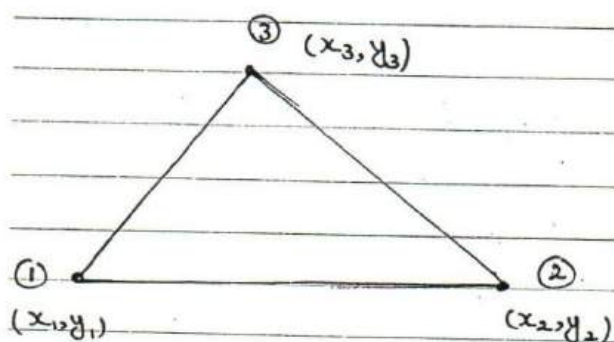
برای هر فایل شبکه المان محدود 2 مشخصه لازم است:

element $\triangle$	node ①	node ②	node ③
①	①	②	③
②	②	④	③
③	③	④	⑤
⋮			
⑩	①	③	⑩

②. Coordinate:

node	X-node	Y-node
①	x_1	y_1
②	x_2	y_2
③	x_3	y_3
⋮	⋮	⋮
⑩	x_{10}	y_{10}

$$\underset{20 \times 20}{\underbrace{\mathbf{K}}}_{20 \times 1} \cdot \underset{20 \times 1}{\underbrace{\mathbf{U}}} = \underset{20 \times 1}{\underbrace{\mathbf{F}}}$$



$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

Date

Subject

⑩

⑤

④

③

②

①

$$u_1 \begin{matrix} \textcircled{1} \\ K_{11} + K_{11} \end{matrix}$$

$$v_1$$

$$u_2 \begin{matrix} \textcircled{1} \\ K_{12} \end{matrix}$$

$$v_2$$

$$u_3 \begin{matrix} \textcircled{1} \\ K_{13} + K_{12} \end{matrix}$$

$$v_3$$

$$u_4$$

$$v_4$$

$$u_5$$

$$v_5$$

$$u_{10} \begin{matrix} \textcircled{10} \\ K_{13} \end{matrix}$$

$$v_{10}$$

$$u_1$$

$$v_1$$

$$u_2$$

$$v_2$$

$$u_3$$

$$v_3$$

$$u_4$$

$$v_4$$

$$u_5$$

$$v_5$$

$$u_1$$

$$v_1$$

$$K_{21}$$

$$K_{22} + K_{11}$$

$$K_{23} + K_{13}$$

$$K_{12}$$

$$K_{31} + K_{21}$$

$$K_{32} + K_{31}$$

$$K_{33} + K_{33} + K_{11} + K_{22}$$

$$K_{32} + K_{12}$$

$$K_{13}$$

$$K_{23}$$

$$K_{21}$$

$$K_{23} + K_{21}$$

$$K_{22} + K_{22}$$

$$K_{23}$$

$$K_{31}$$

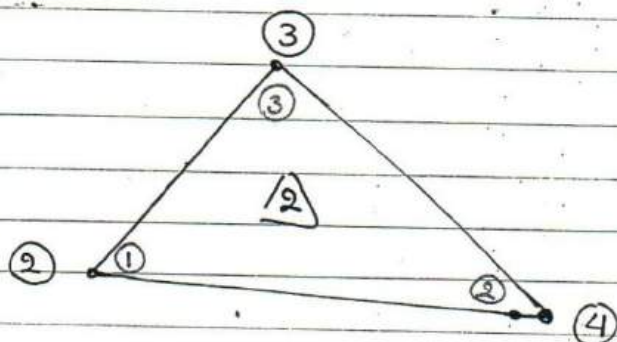
$$K_{32}$$

$$K_{33}$$

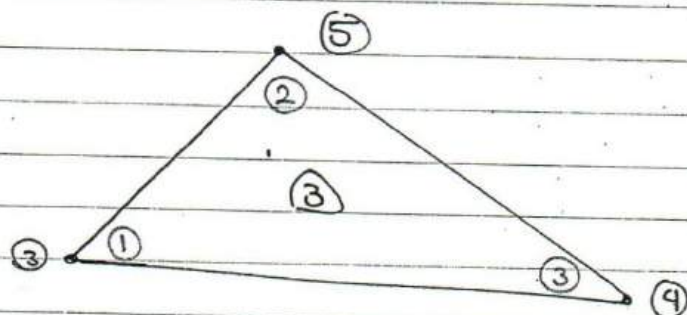
$$K_{31}$$

$$K_{32}$$

$$K_{33}$$



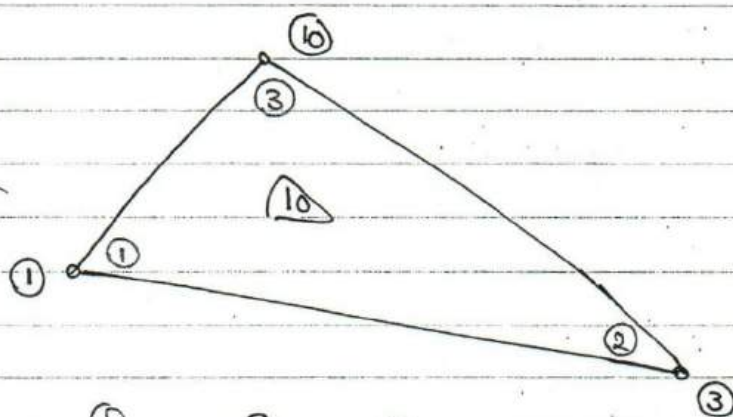
$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{matrix} \textcircled{1} & \textcircled{2} \\ \textcircled{2} & \textcircled{4} \\ \textcircled{3} & \textcircled{3} \end{matrix}$$



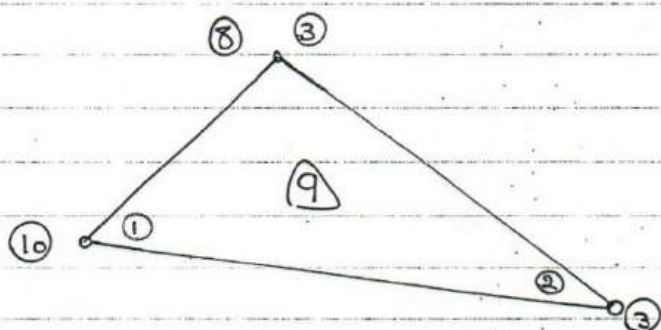
$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{matrix} \textcircled{1} & \textcircled{3} \\ \textcircled{2} & \textcircled{4} \\ \textcircled{3} & \textcircled{5} \end{matrix}$$

[illegible]

Subject Date



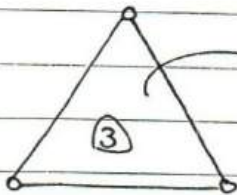
$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{matrix} 1 & 1 \\ 2 & 3 \\ 3 & 10 \end{matrix}$$



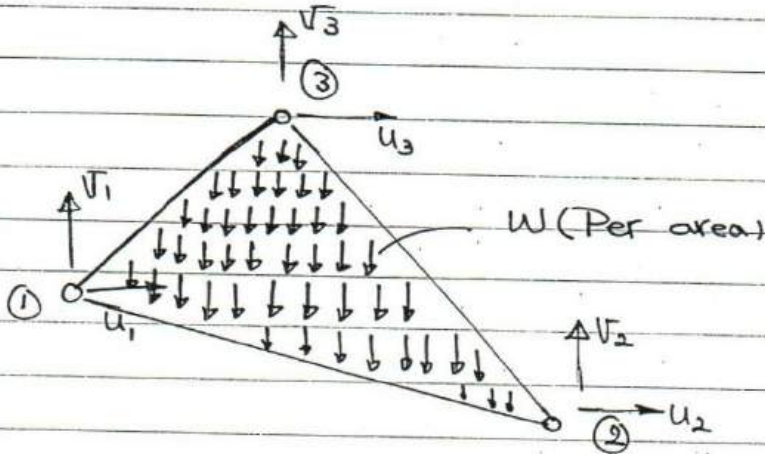
$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{matrix} 1 & 3 \\ 2 & 8 \\ 3 & 10 \end{matrix}$$

$$K \cdot u = F$$

$$F_c + F_b + F_q + F_e + F_v$$



درق به در W بر واحد سطح



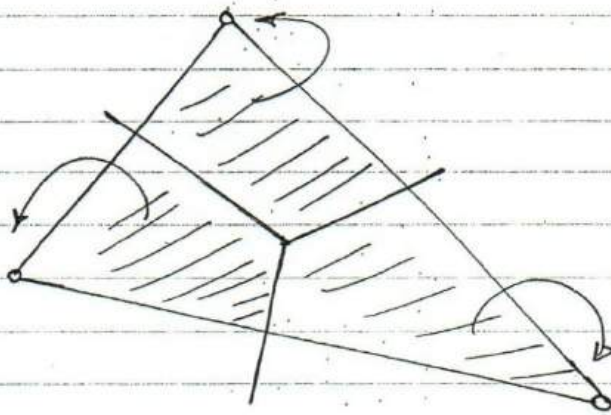
$$F_b = \int_r N^T \cdot b \cdot dV \quad \rightarrow \quad b = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -w \end{Bmatrix}$$

$$F_b = \int_r N^T \cdot b \cdot dV = \begin{Bmatrix} F_b^{(1)} \\ F_b^{(2)} \\ F_b^{(3)} \end{Bmatrix} = \begin{Bmatrix} F_{bx}^{(1)} \\ F_{by}^{(1)} \\ F_{bx}^{(2)} \\ F_{by}^{(2)} \\ F_{bx}^{(3)} \\ F_{by}^{(3)} \end{Bmatrix}$$

$\int_r N_1(-w) dV_{(x,y)}$
 $-\int_r N_2(-w) dV_{(x,y)}$
 $\int_r N_3(-w) dV_{(x,y)}$

$$N_i(x,y) = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$

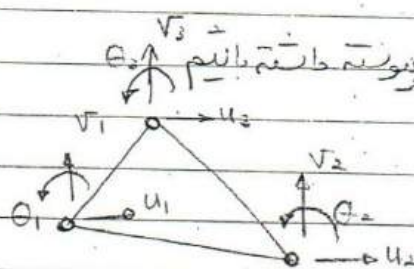
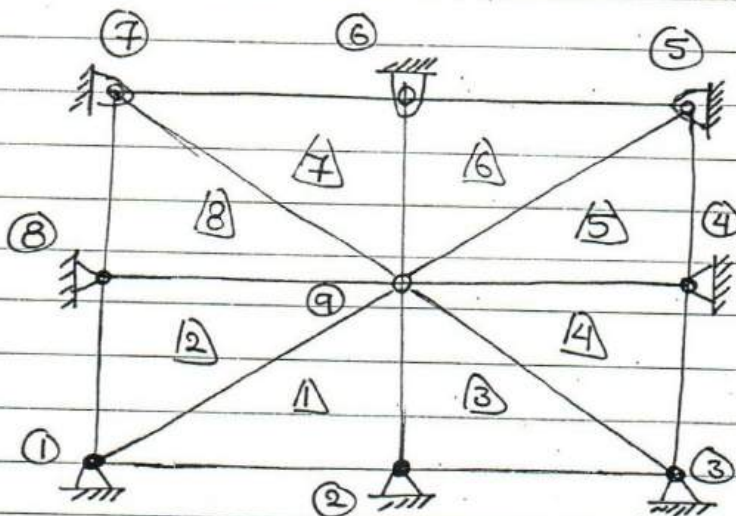
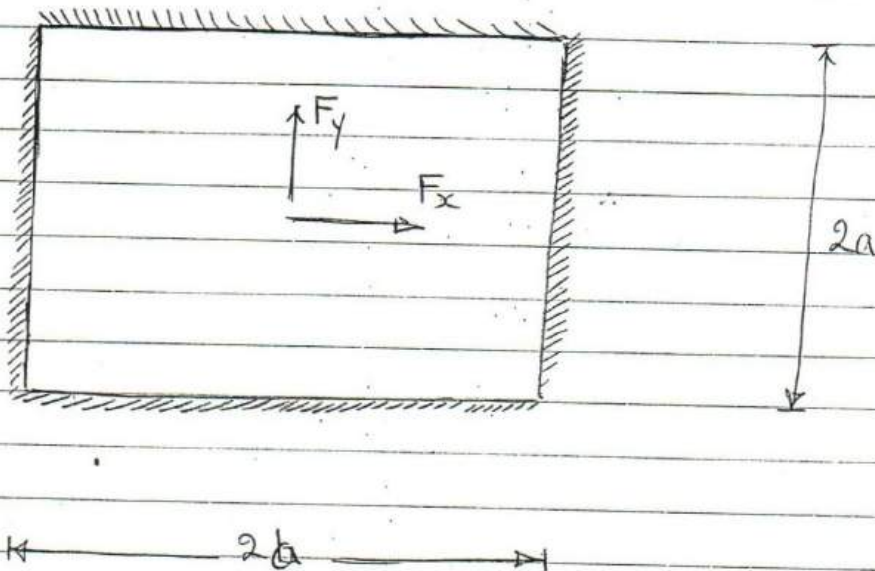
$$\left[\begin{array}{cc} N_1 & \cdot \\ \cdot & N_1 \\ \hline N_2 & \cdot \\ \cdot & N_2 \\ N_3 & \cdot \\ \cdot & N_3 \end{array} \right] \left\{ \begin{array}{c} 0 \\ -w \end{array} \right\} dV$$



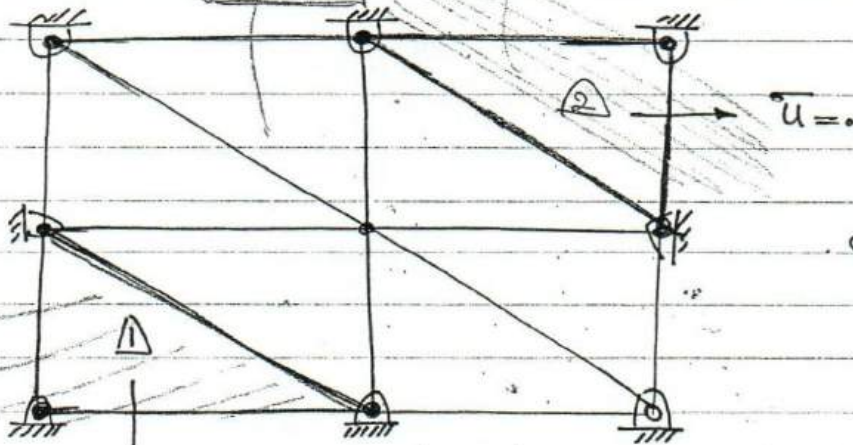
$$\rightarrow F_b = \left\{ \begin{array}{c} 0 \\ -\frac{1}{3} w \cdot A \cdot t \\ \hline 0 \\ -\frac{1}{3} w \cdot A \cdot t \\ 0 \\ -\frac{1}{3} w \cdot A \cdot t \end{array} \right\}$$

مثال: مطلوب است حل و رسم شکل زیر که تحت دو بار مستقر F_x و F_y قرار گرفته است با استفاده از ۸ امان

۳- برای



الو 2 هج جانب کا ہزار سی $\bar{u}_2 = \{0\} \Rightarrow$



این مش بندی غلط است

$$u(\alpha) = \bar{N} \cdot \bar{u} \rightarrow \bar{u}_1 = 0 \rightarrow \bar{u} = \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} \Rightarrow \text{ان را به اصل جانبی آورد}$$

$$K \times \bar{u} = F$$

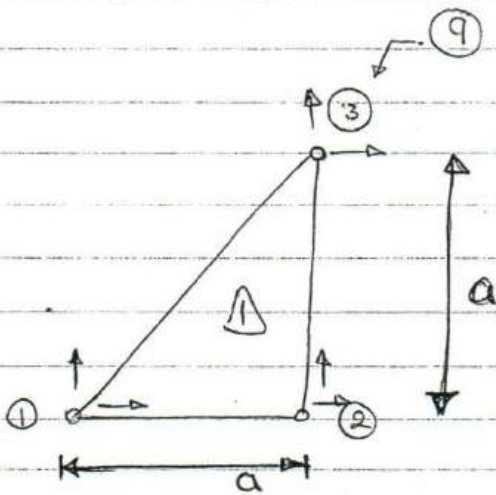
18x18

18x1

18x1

B.Cs

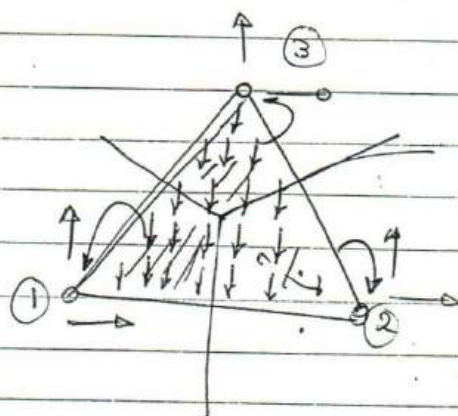
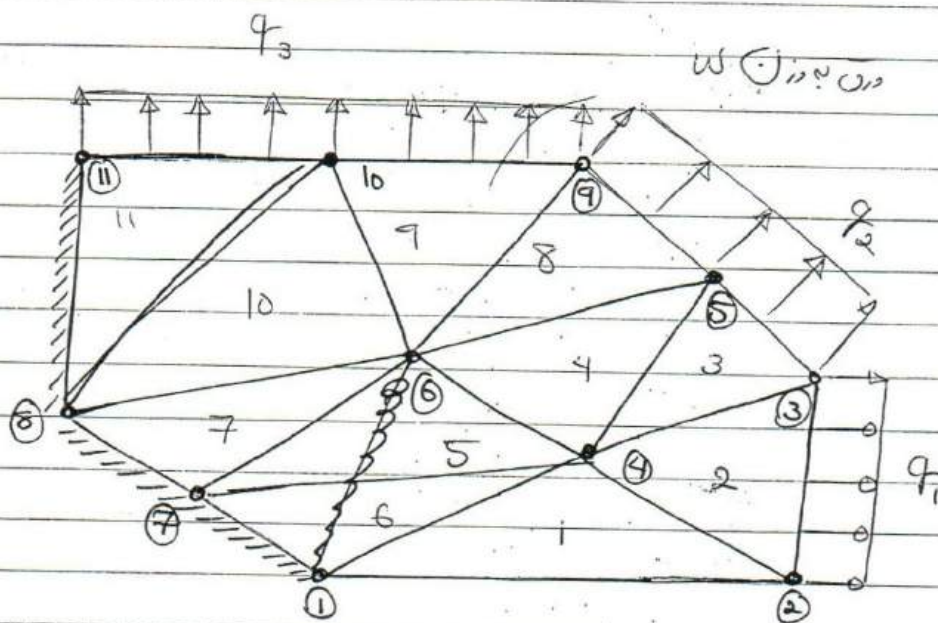
$$\hat{K}_{2 \times 2} \begin{Bmatrix} u_q \\ v_q \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$



$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$$K = \bar{A}^T \cdot \bar{K} \cdot \bar{A} \rightarrow \bar{A} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \Rightarrow \alpha = 180^\circ$$

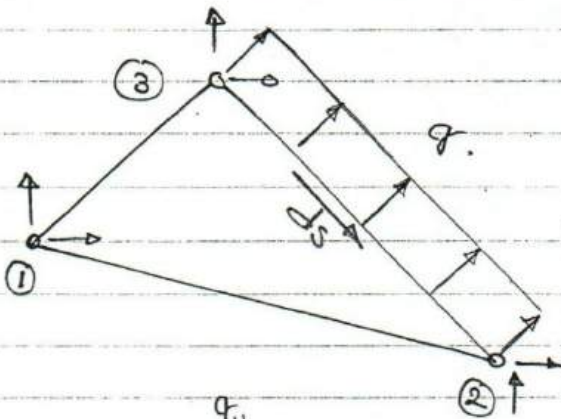
$$K = \bar{A}^T \cdot \bar{K} \cdot \bar{A} \rightarrow \alpha = -90^\circ$$



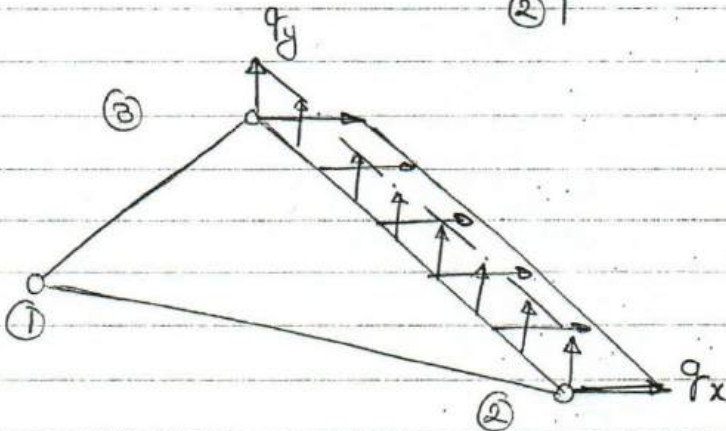
$$F_b = \int_V N^T \cdot b \cdot dV = \begin{Bmatrix} F_{bx}^1 \\ F_{by}^1 \\ F_{bx}^2 \\ F_{by}^2 \\ F_{bx}^3 \\ F_{by}^3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1/3 wAt \\ 0 \\ -1/3 wAt \\ 0 \\ -1/3 wAt \end{Bmatrix}$$

برای معادل خاکی از بارگذاری سطحی

$$F_q = \int \tilde{N}^T \tilde{q} \cdot ds = \begin{Bmatrix} F_{q_1} \\ F_{q_2} \\ F_{q_3} \end{Bmatrix}$$



$$\Rightarrow F_q = \begin{Bmatrix} F_{q_x}^1 \\ F_{q_y}^1 \\ F_{q_x}^2 \\ F_{q_y}^2 \\ F_{q_x}^3 \\ F_{q_y}^3 \end{Bmatrix} \begin{matrix} \rightarrow \int N_2 \cdot q_x \cdot ds \\ \rightarrow \int N_2 \cdot q_y \cdot ds \\ \rightarrow \int N_3 \cdot q_x \cdot ds \\ \rightarrow \int N_3 \cdot q_y \cdot ds \end{matrix}$$

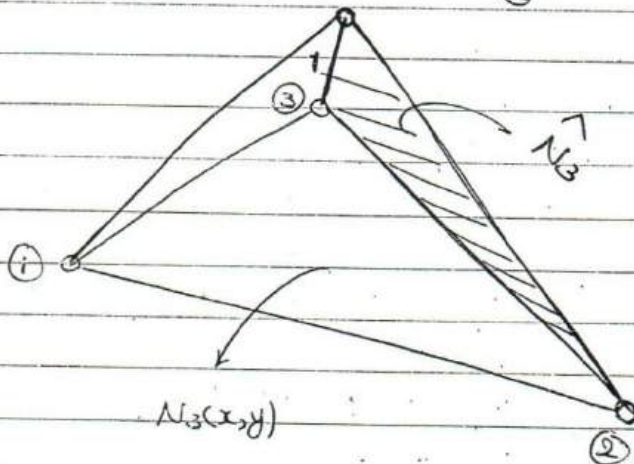
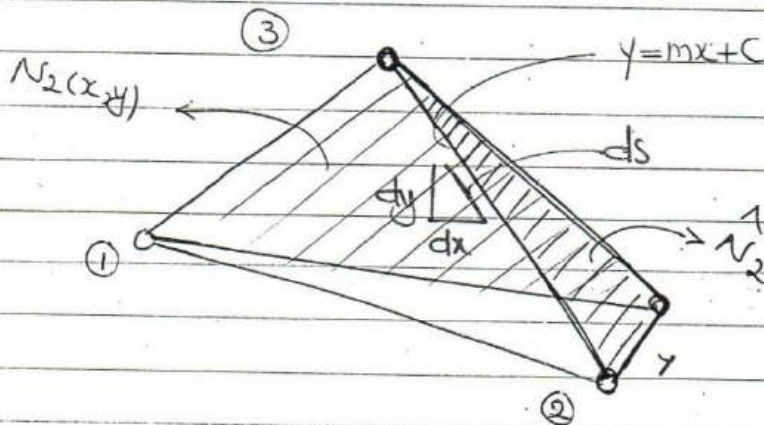


$$F_q = \begin{Bmatrix} 0 \\ 0 \\ \int N_2 q_x ds \\ \int N_2 q_y ds \\ \int N_3 q_x ds \\ \int N_3 q_y ds \end{Bmatrix}$$

$$\rightarrow F_q = \int_S \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_2 \\ 0 & 0 & 0 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} \cdot ds$$

$$N_2(x, y) = \frac{1}{2A} (\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3(x, y) = \frac{1}{2A} (\alpha_3 + \beta_3 x + \gamma_3 y)$$



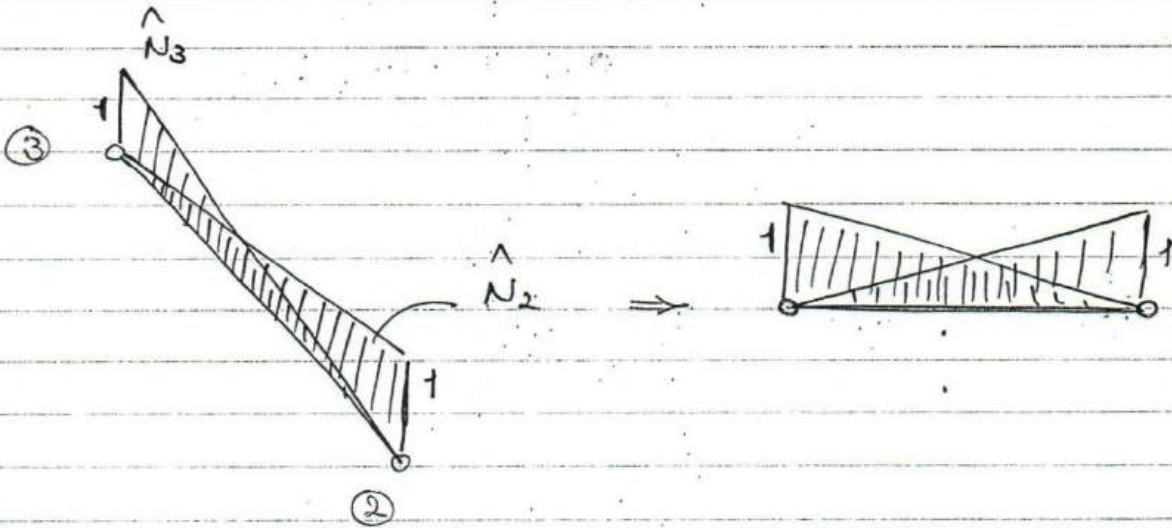
$$\oint F_{q_x}^2 = \int N_2(x, y) q_x ds = \int \hat{N}_2 q_x ds$$

$$\hat{N}_2 = N_2(x, y) \Big|_{y=mx+C}$$

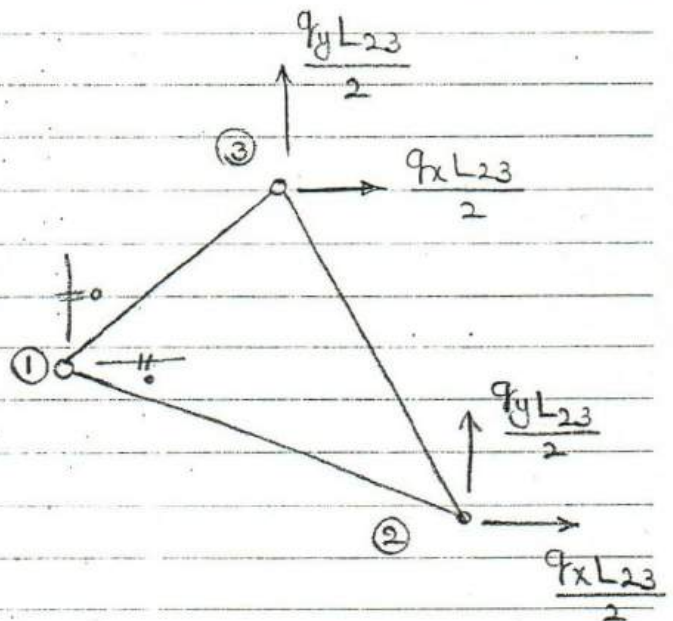
$$F_{q_x}^2 = \int \frac{1}{2A} [\alpha_2 + \beta_2 x + \gamma_2 (mx + C)] \cdot q_x$$

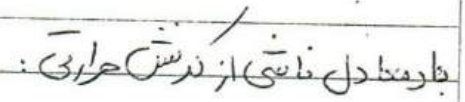
$$ds^2 = dx^2 + dy^2 \rightarrow ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} dx = \sqrt{m^2 + 1} dx$$

$$F_{q_x} = \int_{x_3}^{x_2} \frac{1}{2A} [\alpha_2 + \beta_2 x + \gamma_2 (mx + c)] q_x \sqrt{m^2 + 1} dx = \frac{q_x L_{23}}{2}$$



$$\rightarrow F_q = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ q_x L_{23}/2 \\ q_y L_{23}/2 \\ \vdots \\ q_x L_{23}/2 \\ q_y L_{23}/2 \end{Bmatrix}$$



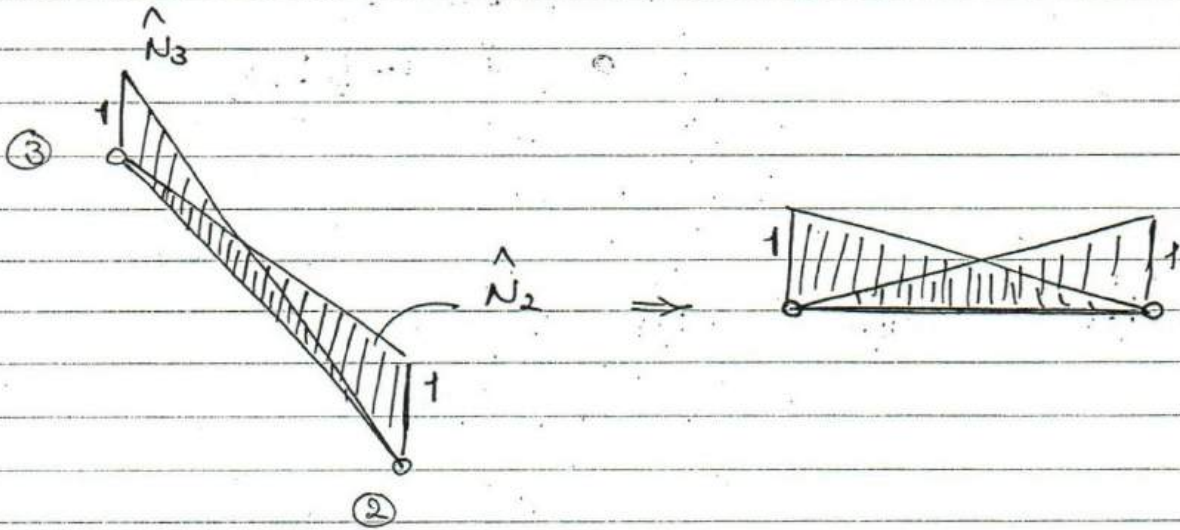


$$F_{\epsilon_0} = \int \underbrace{B^T}_{6 \times 3} \cdot \underbrace{D}_{(3 \times 3)} \cdot \underbrace{\epsilon_0}_{(3 \times 1)} dV = \begin{Bmatrix} F_{\epsilon_0 x}^1 \\ F_{\epsilon_0 y}^1 \\ F_{\epsilon_0 x}^2 \\ F_{\epsilon_0 y}^2 \\ F_{\epsilon_0 x}^3 \\ F_{\epsilon_0 y}^3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} B_1 \cdot \gamma_1 \\ \cdot \gamma_1 B_1 \\ \cdot \gamma_2 B_2 \\ B_2 \cdot \gamma_2 \\ \cdot \gamma_3 B_3 \\ B_3 \cdot \gamma_3 \end{bmatrix} \cdot D \cdot \begin{Bmatrix} \epsilon_{0x} \\ \epsilon_{0y} \\ \gamma_{xy} \end{Bmatrix} \quad t_c$$

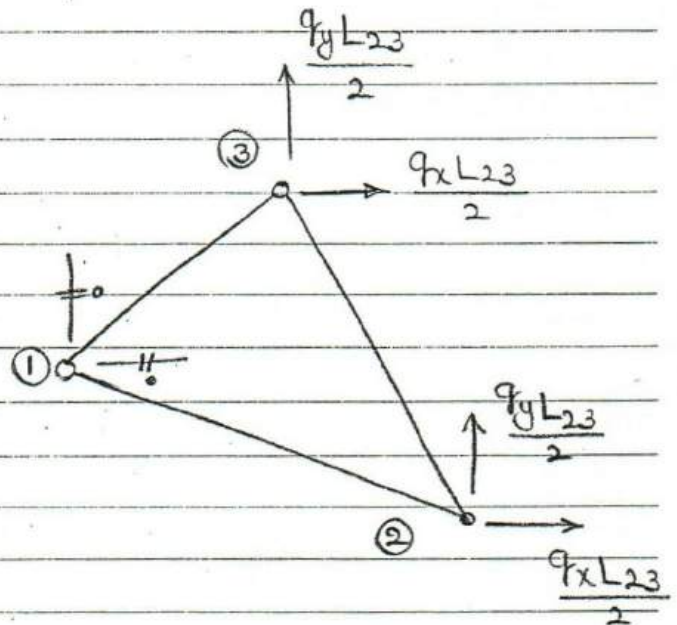


$$ds^2 = dx^2 + dy^2 \rightarrow ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} dx = \sqrt{m^2 + 1} dx$$

$$F_{q_x} = \int_{x_3}^{x_2} \frac{1}{2A} [\alpha_2 + \beta_2 x + \gamma_2 (mx + c)] q_x \sqrt{m^2 + 1} dx = \frac{q_x L_{23}}{2}$$

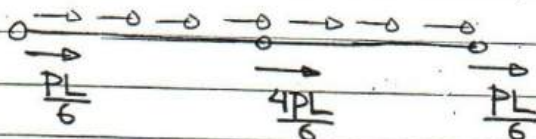
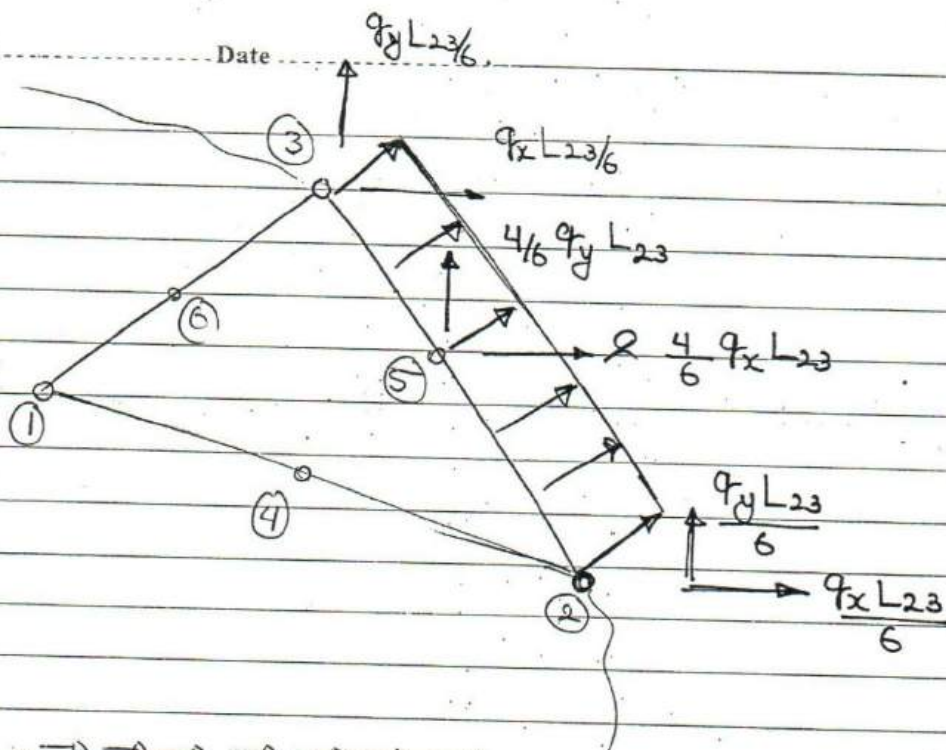


$$\rightarrow F_q = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ q_x L_{23}/2 \\ q_y L_{23}/2 \\ \dots \\ q_x L_{23}/2 \\ q_y L_{23}/2 \end{Bmatrix}$$



Subject

Date



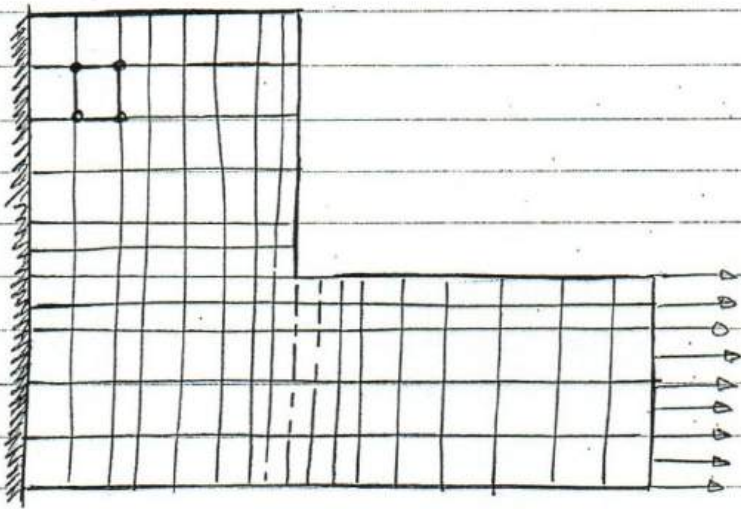
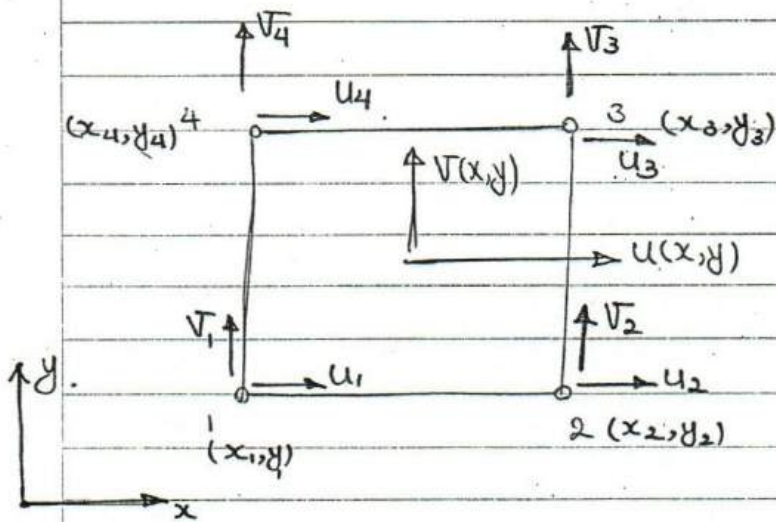
$$\epsilon_o = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

بار معادل ناشی از درش حراری:

$$F_{\epsilon_o} = \int_{\Omega} B^T D \epsilon_o dV = \begin{Bmatrix} F_{\epsilon_o x}^1 \\ F_{\epsilon_o y}^1 \\ F_{\epsilon_o x}^2 \\ F_{\epsilon_o y}^2 \\ F_{\epsilon_o x}^3 \\ F_{\epsilon_o y}^3 \end{Bmatrix} = \int \frac{1}{2A} \begin{bmatrix} B_1 \cdot \gamma_1 \\ \cdot \gamma_1 B_1 \\ \cdot \gamma_2 B_2 \\ \cdot \gamma_2 B_2 \\ B_3 \cdot \gamma_3 \\ \cdot \gamma_3 B_3 \end{bmatrix} \cdot D \cdot \begin{Bmatrix} \epsilon_{ox} \\ \epsilon_{oy} \\ \gamma_{xy} \end{Bmatrix} dt$$

Plane Stress $\rightarrow D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

الماتريks مستطيل - مربعي :



$$u(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$$

$$\rightarrow P(x, y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix}$$

$$v(x, y) = b_0 + b_1 x + b_2 y + b_3 xy$$

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \underline{N}(x, y) \cdot \underline{U}$$

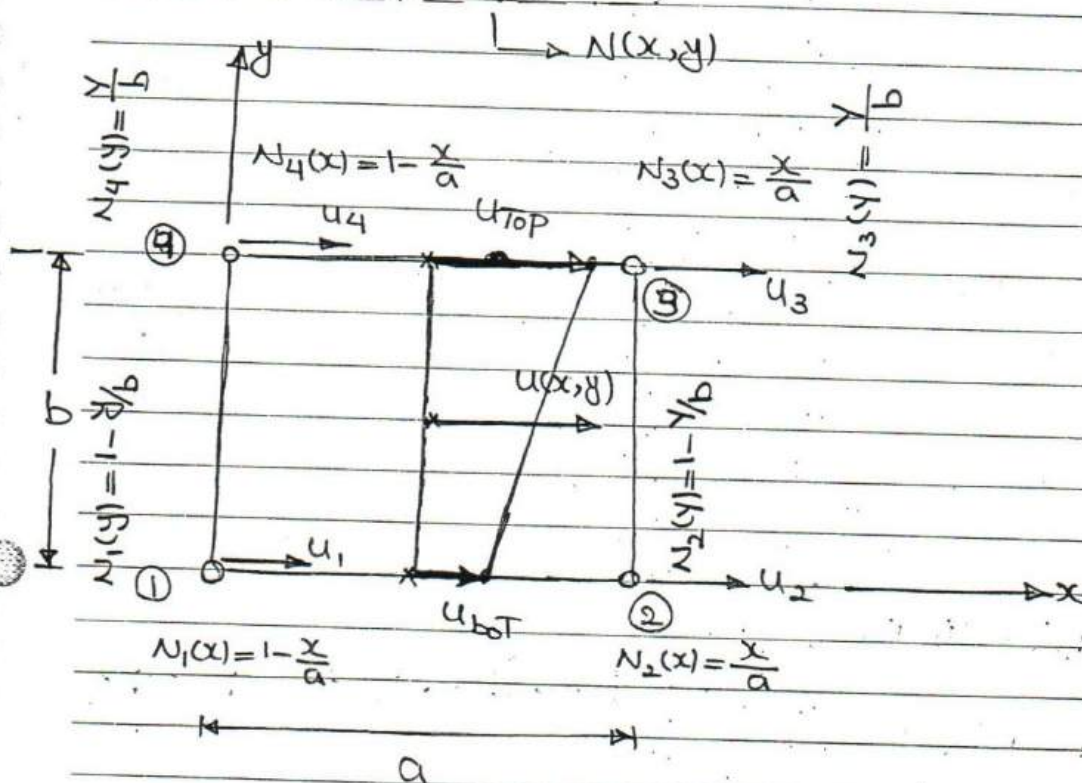
$$\underline{U} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\Rightarrow C_{4 \times 4} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}$$

$$C_{4 \times 4} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}$$

$$u(x, y) = \underbrace{P(x, y)}_{N(x, y)} C_{4 \times 4}^{-1} \underbrace{\bar{U}}_{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}} = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\bar{V}(x, y) = \underbrace{P(x, y)}_{N(x, y)} C_{4 \times 4}^{-T} \underbrace{\bar{U}}_{\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}} = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$



$$u_{bot} \Rightarrow N_1(x) = 1 - \frac{x}{a} \quad N_2(x) = \frac{x}{a}$$

$$u_{bot} = \left(1 - \frac{x}{a}\right) u_1 + \left(\frac{x}{a}\right) u_2$$

$$u_{top} \Rightarrow N_4(y) = 1 - \frac{y}{b} \quad N_3(y) = \frac{y}{b}$$

$$u_{\text{Top}} = \left(1 - \frac{x}{a}\right) u_4 + \left(\frac{x}{a}\right) u_3$$

$$u(x, y) = \left(1 - \frac{y}{b}\right) u_{\text{bot}} + \frac{y}{b} (u_{\text{Top}})$$

$$\rightarrow u(x, y) = \left(1 - \frac{y}{b}\right) \left[\left(1 - \frac{x}{a}\right) u_1 + \left(\frac{x}{a}\right) u_2 \right] + \frac{y}{b} \left[\left(1 - \frac{x}{a}\right) u_4 + \left(\frac{x}{a}\right) u_3 \right]$$

$$\rightarrow u(x, y) = \underbrace{\left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right)}_{N_1(x, y)} u_1 + \underbrace{\left(\frac{x}{a}\right) \left(1 - \frac{y}{b}\right)}_{N_2(x, y)} u_2 + \underbrace{\left(\frac{x}{a}\right) \left(\frac{y}{b}\right)}_{N_3(x, y)} u_3 + \underbrace{\left(1 - \frac{x}{a}\right) \left(\frac{y}{b}\right)}_{N_4(x, y)} u_4$$

$N_1(x) \leftarrow \left(1 - \frac{x}{a}\right)$ $N_1(y) \leftarrow \left(1 - \frac{y}{b}\right)$
 $N_2(x) \leftarrow \left(\frac{x}{a}\right)$ $N_2(y) \leftarrow \left(1 - \frac{y}{b}\right)$
 $N_3(x) \leftarrow \left(\frac{x}{a}\right)$ $N_3(y) \leftarrow \left(\frac{y}{b}\right)$
 $N_4(x) \leftarrow \left(1 - \frac{x}{a}\right)$ $N_4(y) \leftarrow \left(\frac{y}{b}\right)$

$$a = x_2 - x_1 = x_3 - x_4$$

$$b = (y_4 - y_1) = (y_3 - y_2)$$

$$N_i(x, y) = N_i(x) N_i(y) = L_i(x) \cdot L_i(y)$$

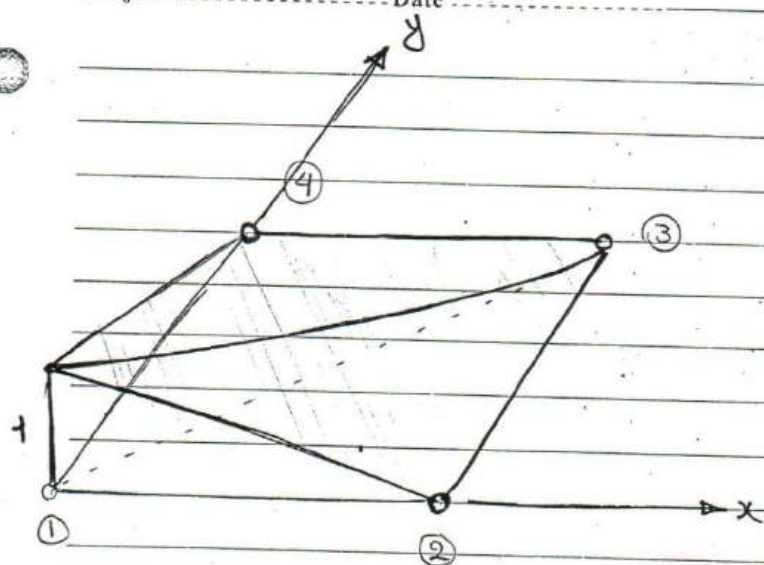
$$N_1(x, y) = \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) = N_1(x) \cdot N_1(y) = L_1(x) \cdot L_1(y)$$

$$N_2(x, y) = \left(\frac{x}{a}\right) \left(1 - \frac{y}{b}\right) = N_2(x) N_2(y) = L_2(x) \cdot L_2(y)$$

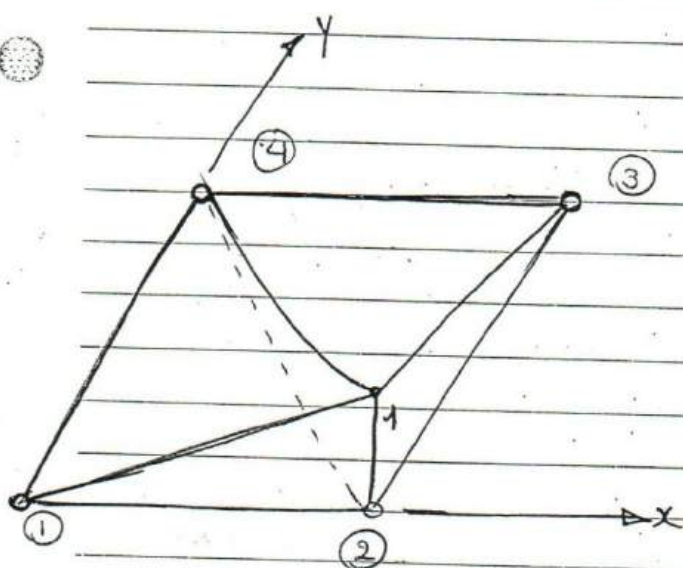
$$N_3(x, y) = \left(\frac{x}{a}\right) \left(\frac{y}{b}\right) = N_3(x) \cdot N_3(y) = L_3(x) \cdot L_3(y)$$

$$N_4(x, y) = \left(1 - \frac{x}{a}\right) \left(\frac{y}{b}\right) = N_4(x) \cdot N_4(y) = L_4(x) \cdot L_4(y)$$

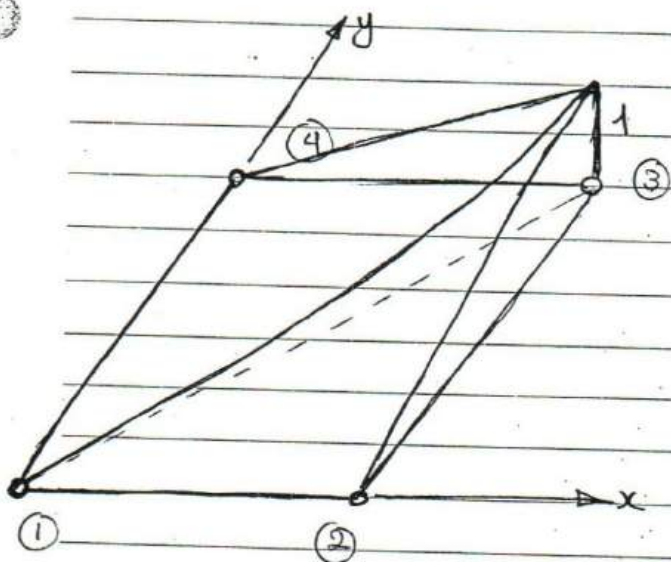
Subject..... Date.....



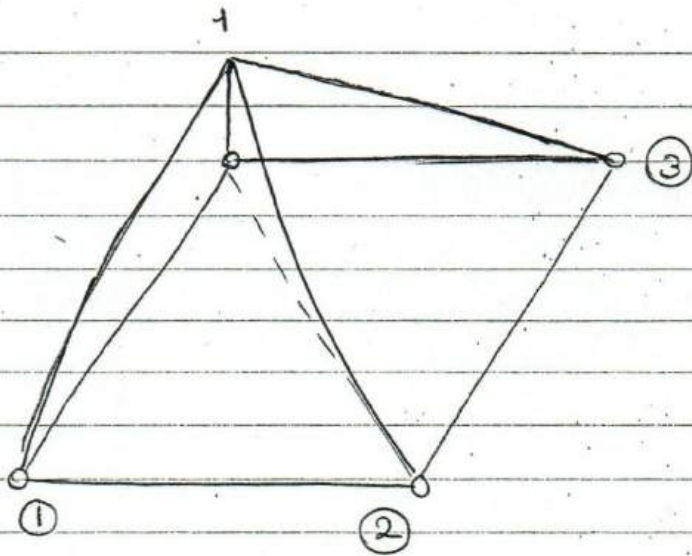
$N_1(x, y)$



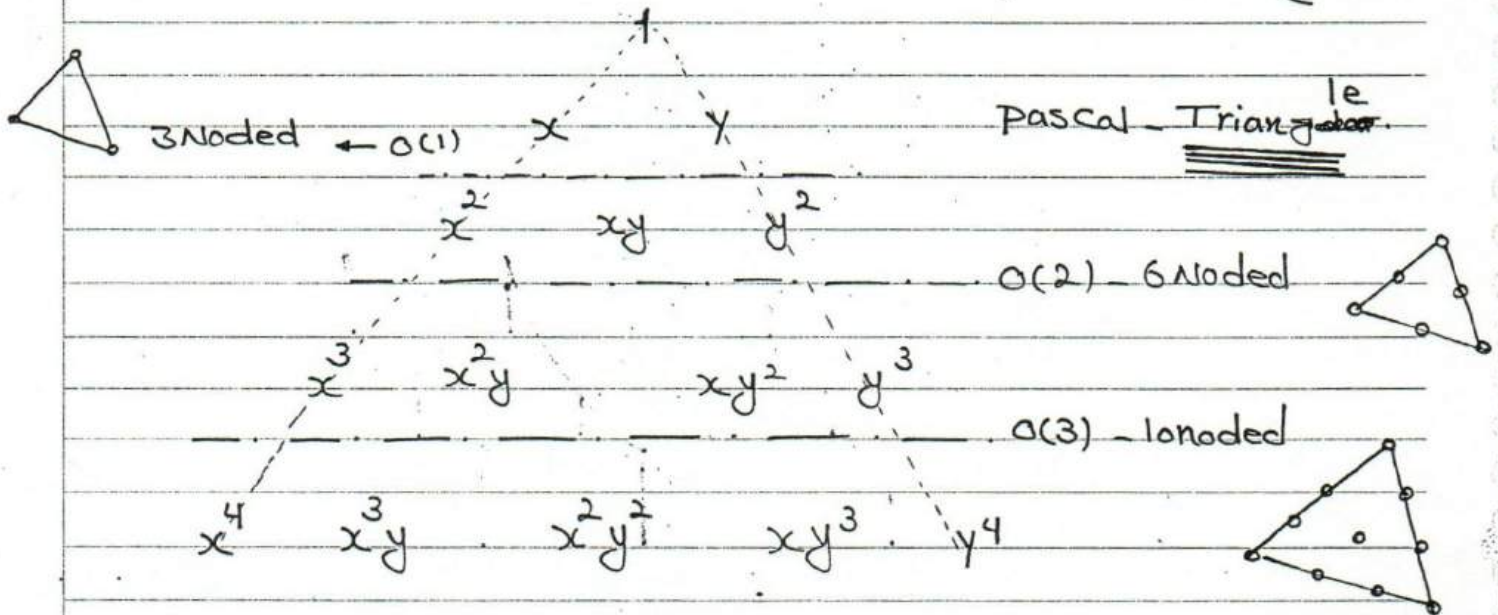
$N_2(x, y)$



$N_3(x, y)$

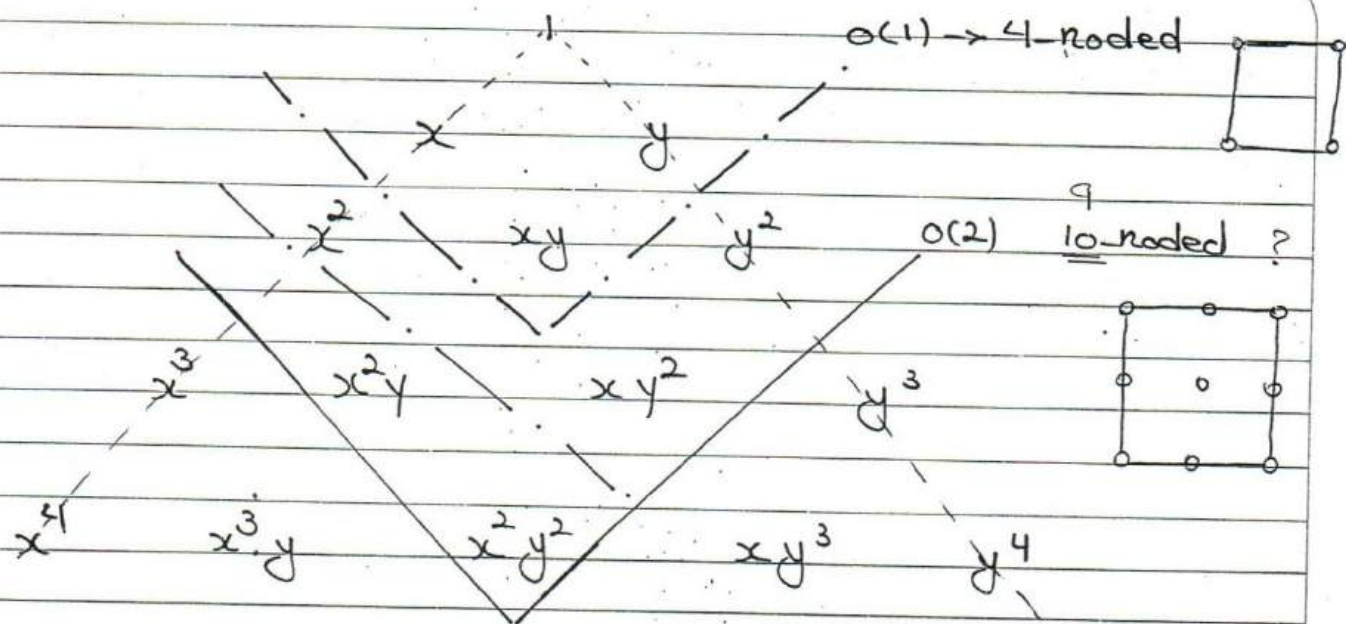


المان مربع مستطیل مرتبه دوم:

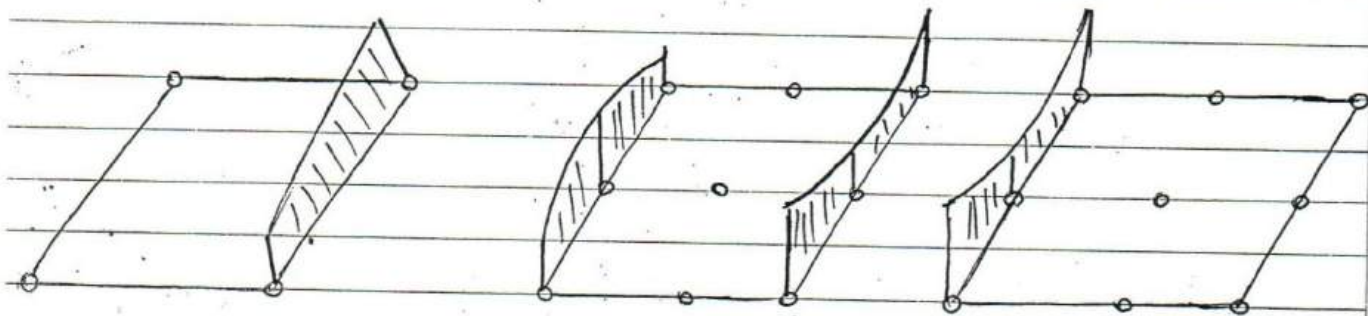


شش پاسکال مختلف المان شش است

برای مربع اگر خواستیم شش خیم پاسکال استفاده کنیم باید اصطلاحاتی انجام بدهیم



$$P(x, y) = \langle 1 \ x \ y \ x^2 \ xy \ y^2 \ x^3y \ xy^2 \ x^2y^2 \rangle$$



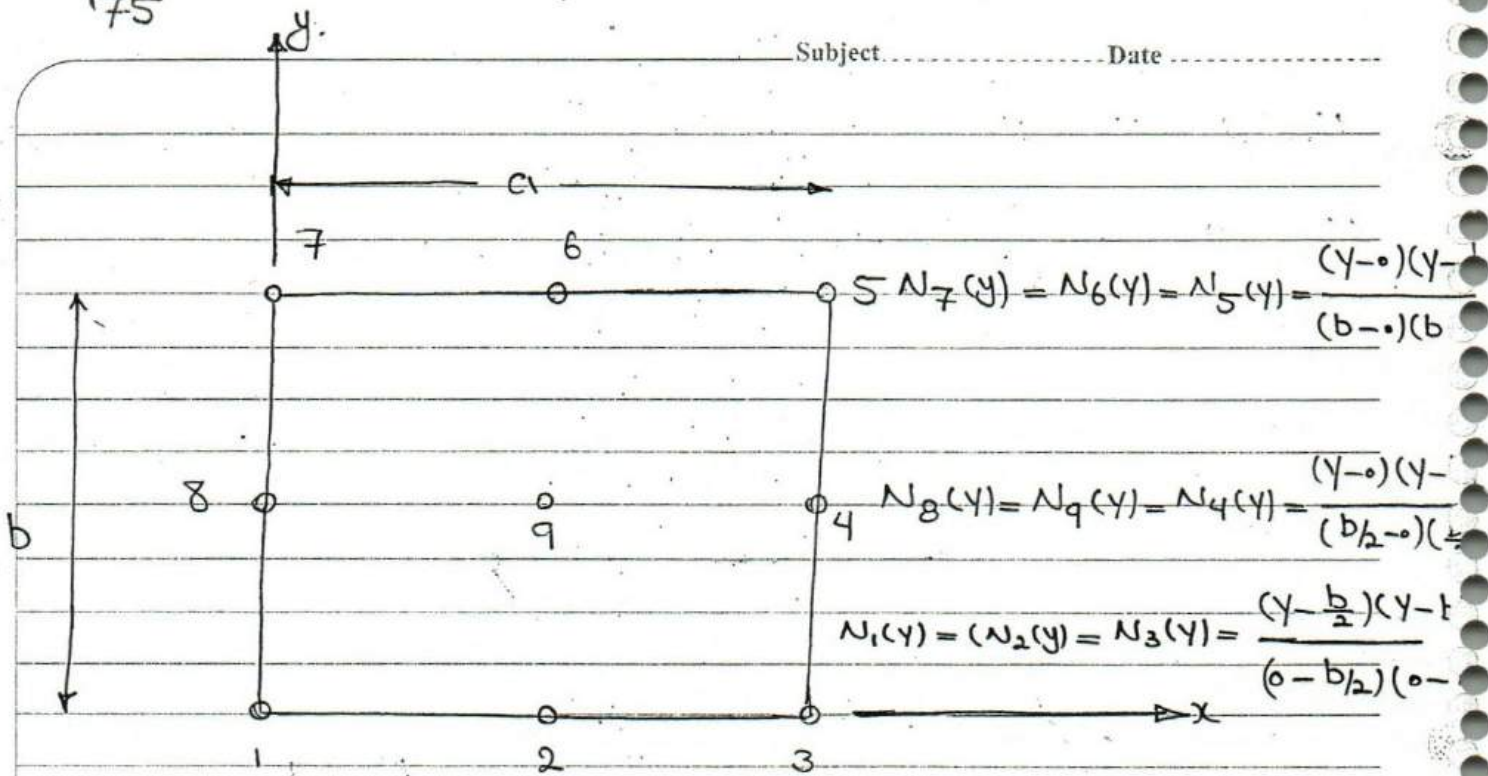
یک المان 4 گرهی داریم و اینم باید المان 4 گرهی اسمبل کنیم چون تابع شکل 4 گرهی خطی است

و بی المان 4 گرهی غیر خطی است
تابع شکل

در هر شرایطی می توان از المان استفاده کرد ← باید المان حامل باشد

منظور از المان حامل این است که انتگرال المان در لبه ها (همچون x و y)

باید هم مرتبه باشند



$$N_1(x) = N_8(x) = N_7(x) =$$

$$N_2(x) = N_9(x) = N_6(x)$$

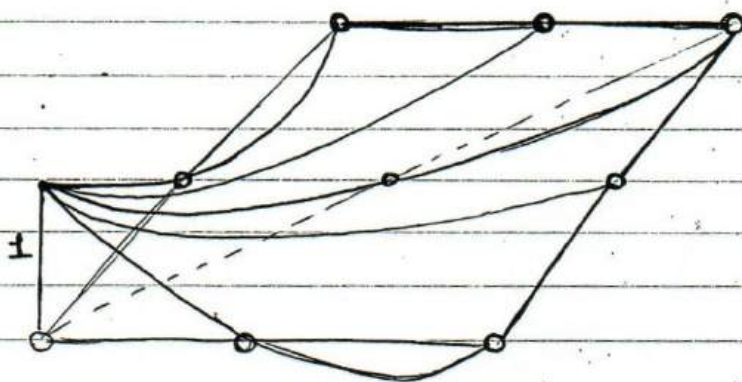
$$N_3(x) = N_4(x) = N_5(x)$$

$$= \frac{(x - \frac{a}{2})(x - a)}{(0 - \frac{a}{2})(0 - a)}$$

$$= \frac{(x - 0)(x - a)}{(\frac{a}{2} - 0)(\frac{a}{2} - a)}$$

$$= \frac{(x - 0)(x - \frac{a}{2})}{(a - 0)(a - \frac{a}{2})}$$

$$N_1(x, y) = N_1(x) \cdot N_1(y) = \frac{4}{a^2 b^2} (x - \frac{a}{2})(x - a)(y - \frac{b}{2})(y - b)$$

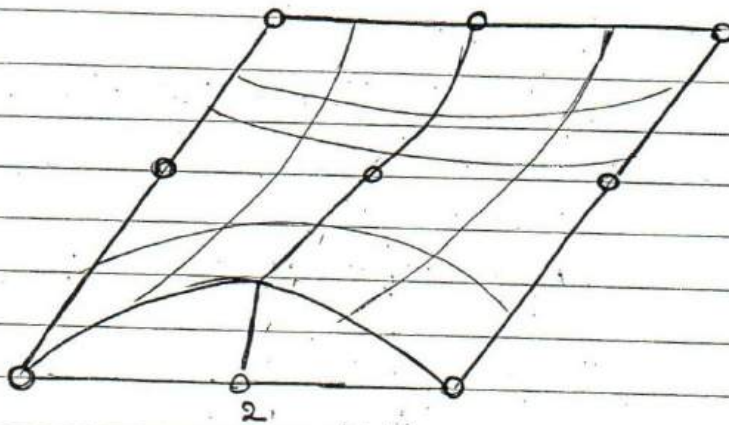


$$N_1(x, y)$$

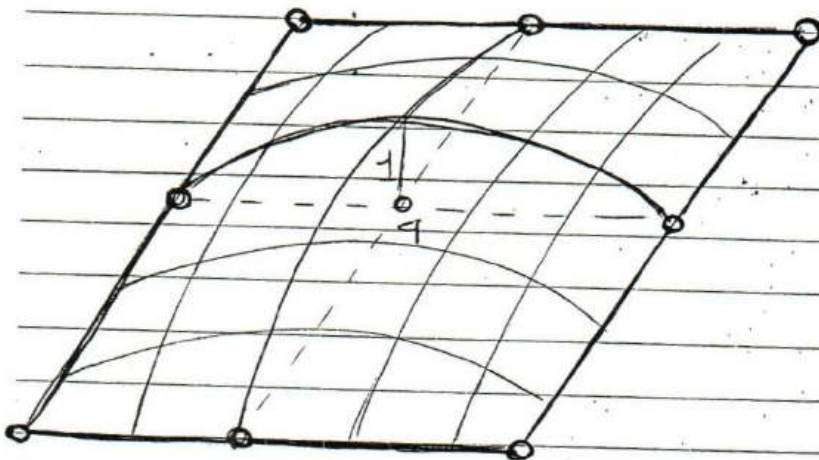
Subject Date

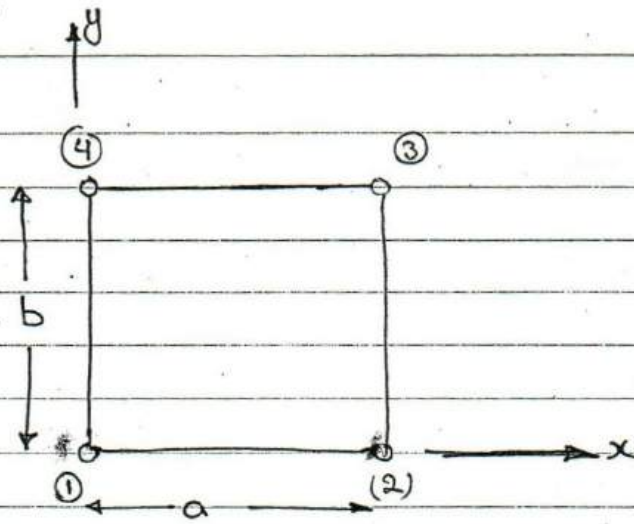
$$N_2(x,y) = N_2(x) N_2(y)$$

$$N_2(x,y) = -\frac{8}{a^2 b^2} x \cdot (x-a) \left(y - \frac{b}{2}\right) (y-b)$$



$$N_4(x,y) = N_4(x) \cdot N_4(y) = \frac{16}{a^2 b^2} x \cdot y \cdot (x-a) (y-b)$$





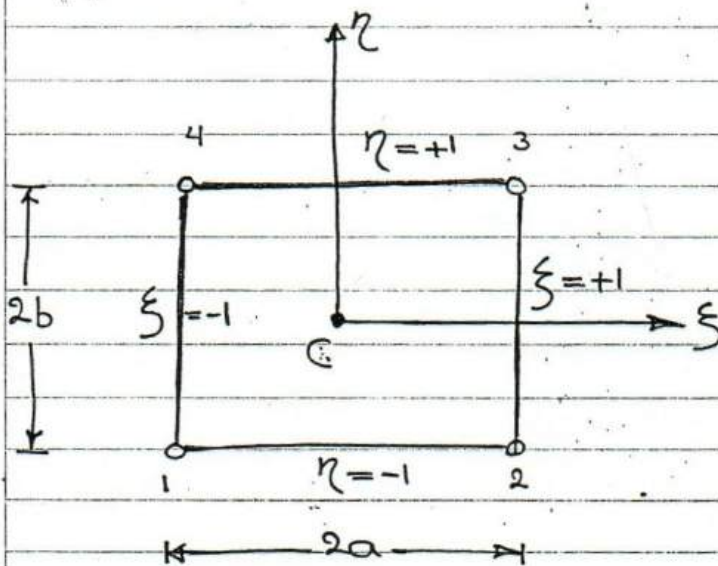
$$N_1(x, y) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$N_3(x, y) = \left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$$

$$N_2(x, y) = \left(\frac{x}{a}\right)\left(1 - \frac{y}{b}\right)$$

$$N_4(x, y) = \left(1 - \frac{x}{a}\right)\left(\frac{y}{b}\right)$$

مختصات في العنصر:



$$-1 \leq \xi, \eta \leq +1$$

$$\xi = \frac{x - x_c}{a}$$

$$\eta = \frac{y - y_c}{b}$$

① نقطة $\rightarrow \begin{cases} \xi = -1 \\ \eta = -1 \end{cases}$

② نقطة $\rightarrow \begin{cases} \xi = +1 \\ \eta = -1 \end{cases}$

③ نقطة $\rightarrow \begin{cases} \xi = +1 \\ \eta = +1 \end{cases}$

④ نقطة $\rightarrow \begin{cases} \xi = -1 \\ \eta = +1 \end{cases}$

$$N_1(\xi, \eta) = \frac{1}{4} (1 - \xi)(1 - \eta)$$

$$N_2(\xi, \eta) = \frac{1}{4} (1 + \xi)(1 - \eta)$$

$$N_3(\xi, \eta) = \frac{1}{4} (1 + \xi)(1 + \eta)$$

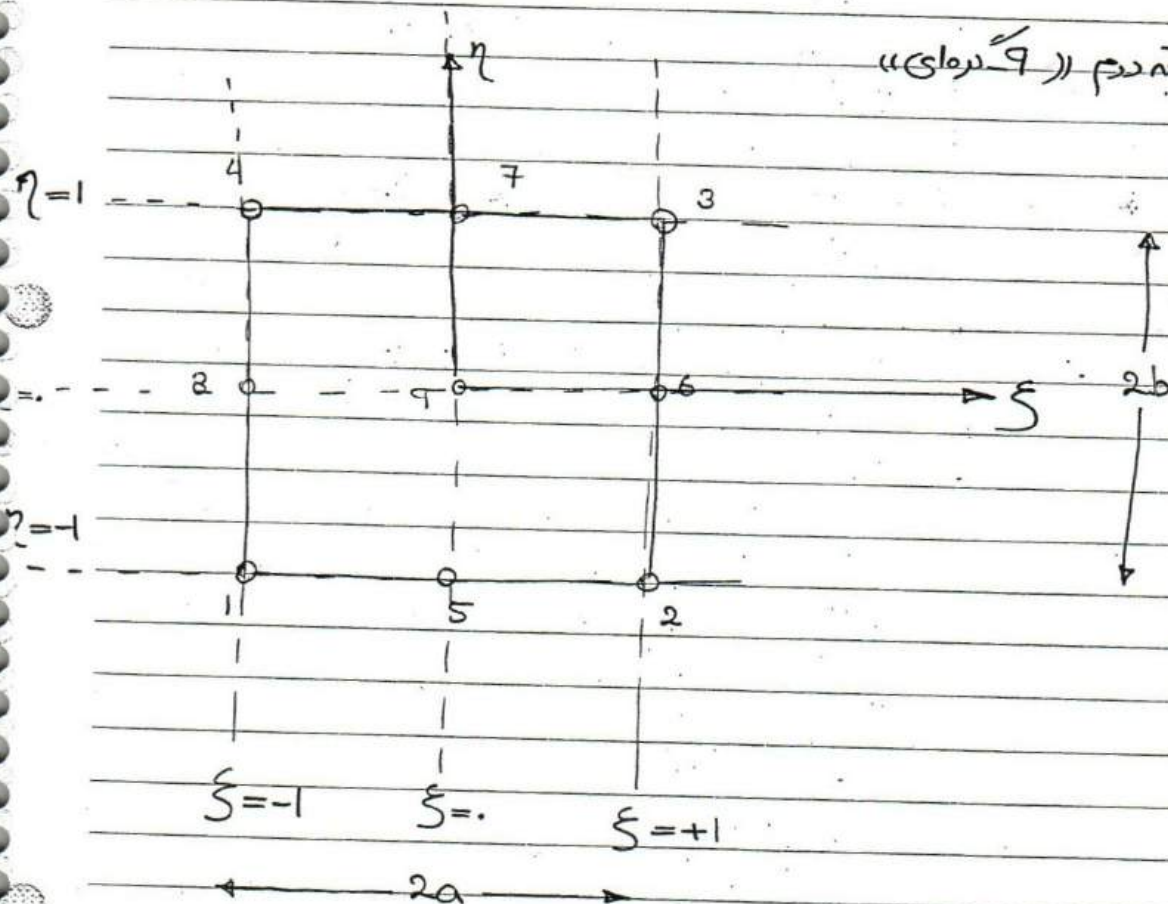
$$N_4(\xi, \eta) = \frac{1}{4} (1 - \xi)(1 + \eta)$$

مقدار دارد

مقدار دارد

$$N_i(\xi, \eta) = \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i \eta) \quad i = 1, 2, 3, 4$$

المان ۴-جوش مربعه درم «۹-گره ای»



$$N_1(\xi, \eta) = \frac{1}{4} \xi(1-\xi)\eta(1-\eta)$$

$$N_i(\xi, \eta) = N_i(\xi) \otimes N_i(\eta)$$

$$P(x, y) = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & x^2y & xy^2 & x^2y^2 \end{pmatrix}$$

$\xrightarrow{\eta} \xi \quad \xrightarrow{\xi} \xi\eta \quad \xrightarrow{\eta^2} \eta^2 \quad \xrightarrow{\xi\eta^2} \xi\eta^2$

$$N_2(\xi, \eta) = \frac{1}{4} \xi(1+\xi)\eta(1-\eta)$$

$\xrightarrow{-\frac{1}{4} \eta}$

$$N_5(\xi, \eta) = -\frac{1}{2} (1+\xi)(1-\xi)\eta(1-\eta)$$

$$N_9(\xi, \eta) = 1(1-\xi)(1+\xi)(1+\eta)(1-\eta) = (1-\xi^2)(1-\eta^2)$$

در این روش هم نه منحنی از خطوط در داخل المان هندسه را قطع می کنند

محل برخورد خطوط در داخل المان منطبق بر نره (Node) ها است

محاسبات مربوط به المان چهارگونی: «ماتریس سختی و بردار نیروها»

$$\underline{\epsilon} = \underline{B} \cdot \underline{u} \rightarrow \underline{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \sum N_i u_i \\ \sum N_i v_i \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \sum_{i=1}^4 \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$\underline{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \sum_{i=1}^4 \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$(x, y) \leftrightarrow (\xi, \eta)$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{1}{a} \frac{\partial N_i}{\partial \xi}$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{1}{b} \frac{\partial N_i}{\partial \eta}$$

$$B_i = \begin{bmatrix} \frac{1}{a} \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{1}{b} \frac{\partial N_i}{\partial \eta} \\ \frac{1}{b} \frac{\partial N_i}{\partial \eta} & \frac{1}{a} \frac{\partial N_i}{\partial \xi} \end{bmatrix}$$

$$\frac{\partial N_i}{\partial \xi} = \frac{1}{4} \xi_i (1 + \eta_i \eta)$$

$$\frac{\partial N_i}{\partial \eta} = \frac{1}{4} \eta_i (1 + \xi_i \xi)$$

$$K_{ij} = \int \tilde{B}_i^T \cdot \tilde{D} \cdot \tilde{B}_j \, dV \rightarrow t \cdot dA$$



$$K_{(8 \times 8)} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$$

Plane Stress:

$$K_{ij} = \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} \frac{1}{a} \frac{\partial N_i}{\partial \xi} & 0 & \frac{1}{b} \frac{\partial N_i}{\partial \eta} \\ 0 & \frac{1}{b} \frac{\partial N_i}{\partial \eta} & \frac{1}{a} \frac{\partial N_i}{\partial \xi} \end{bmatrix} \cdot \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} d\xi d\eta$$

$$\begin{bmatrix} \frac{1}{a} \frac{\partial N_j}{\partial \xi} & 0 \\ 0 & \frac{1}{b} \frac{\partial N_j}{\partial \eta} \\ \frac{1}{b} \frac{\partial N_j}{\partial \eta} & \frac{1}{a} \frac{\partial N_j}{\partial \xi} \end{bmatrix} t(a \cdot b) d\xi \cdot d\eta$$

$$dA = dx \cdot dy = (a \cdot d\xi)(b \cdot d\eta)$$

K=8x8

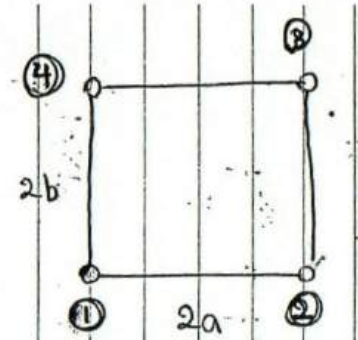
$$E, t$$

$$12(1-\nu^2)$$

Sym

u_1	u_2	u_3	u_4	u_5
$\frac{4}{r} + 4pr$	$-\frac{4}{r} + 2pr$	$-\lambda$	$-\frac{2}{r} - 2pr$	$-\mu$
$4r + \frac{4p}{r}$	λ	$2r - \frac{4p}{r}$	$-\mu$	$-2r - \frac{2p}{r}$
$\frac{4}{r} + 4pr$	$-\mu$	$\frac{2}{r} - 4pr$	$-\lambda$	$-\frac{2}{r} - 2pr$
$4r + \frac{4p}{r}$	λ	$-4r + \frac{2p}{r}$	μ	$-2r - \frac{2p}{r}$
$\frac{4}{r} + 4pr$	μ	$-\frac{4}{r} + 2pr$	$-\lambda$	
$4r + \frac{4p}{r}$	λ	$2r - \frac{4p}{r}$		
$\frac{4}{r} + 4pr$	$-\mu$			

u_1
 u_2
 u_3
 u_4
 u_5



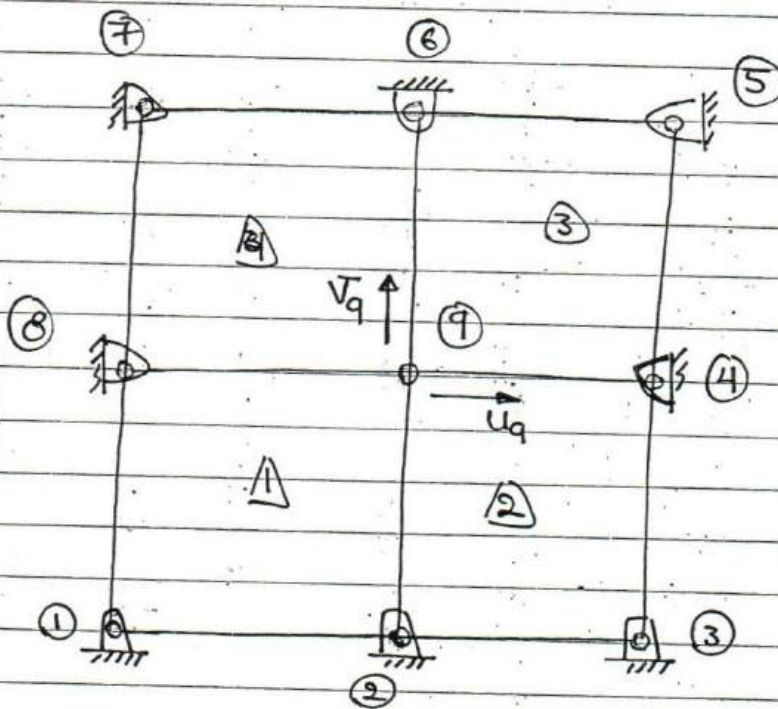
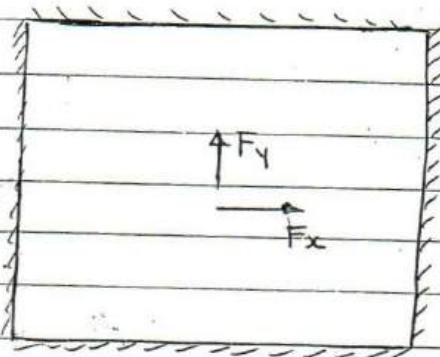
$$r = \frac{a}{b}$$

$$\rho = \frac{1-\nu}{2}$$

$$\mu = \frac{3}{2}(1+\nu)$$

$$\lambda = \frac{3}{2}(1-3\nu)$$

مطلوب است حل مثال قبل با استفاده از ۴ المان کمره



$$K_{18 \times 18} \cdot \bar{U}_{18 \times 1} = F_{18 \times 1} \xrightarrow{\text{B.C.s}} K_{2 \times 2} \begin{Bmatrix} u_q \\ v_q \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

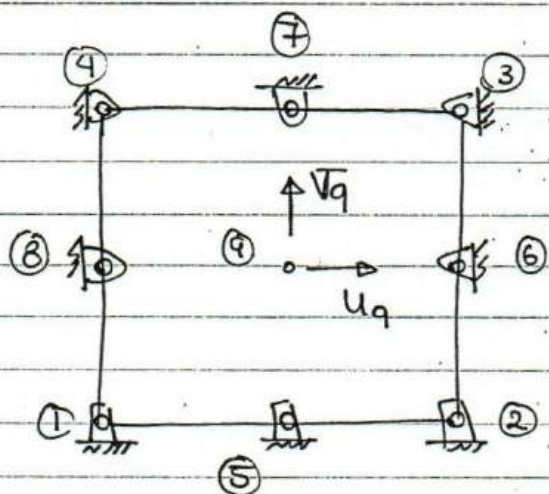
$$K_{33} + K_{99} + K_{11} + K_{22}$$

$$4 \begin{bmatrix} \frac{4}{r} + 4pr & 0 \\ 0 & 4r + \frac{4p}{r} \end{bmatrix} \begin{Bmatrix} u_q \\ v_q \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

$$u_q = \frac{F_x}{\frac{16}{r} + 16pr}$$

$$v_q = \frac{F_y}{16r + \frac{16p}{r}}$$

آر سوال قبل واپس المان کے روش سے بھی حل کیا گیا۔



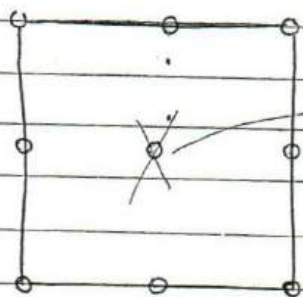
$$\hat{K}_{2 \times 2} \begin{Bmatrix} u_q \\ v_q \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

$$K_{ij} = \int \tilde{B}_i^T \cdot \tilde{D} \cdot \tilde{B}_j \cdot t \cdot dA \quad i, j = 1, 2, \dots, 9$$

$$K_{qq} = \int \tilde{B}_q^T \cdot \tilde{D} \cdot \tilde{B}_q \cdot t \cdot dA \quad i=q \quad j=q$$

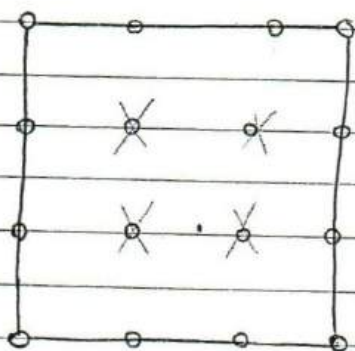
$$B_q = \begin{bmatrix} \frac{1}{a} \frac{\partial N_q}{\partial \xi} & -2\xi(1-\eta^2) \\ \frac{1}{b} \frac{\partial N_q}{\partial \eta} & -2\eta(1-\xi^2) \\ \frac{1}{b} \frac{\partial N_q}{\partial \eta} & \frac{1}{a} \frac{\partial N_q}{\partial \xi} \end{bmatrix} \rightarrow -2\eta(1-\xi^2)$$

$N_q(\xi, \eta) = (1-\xi^2)(1-\eta^2)$



امنیت این طرح را به صورت زیر
امنیت این طرح را به صورت زیر

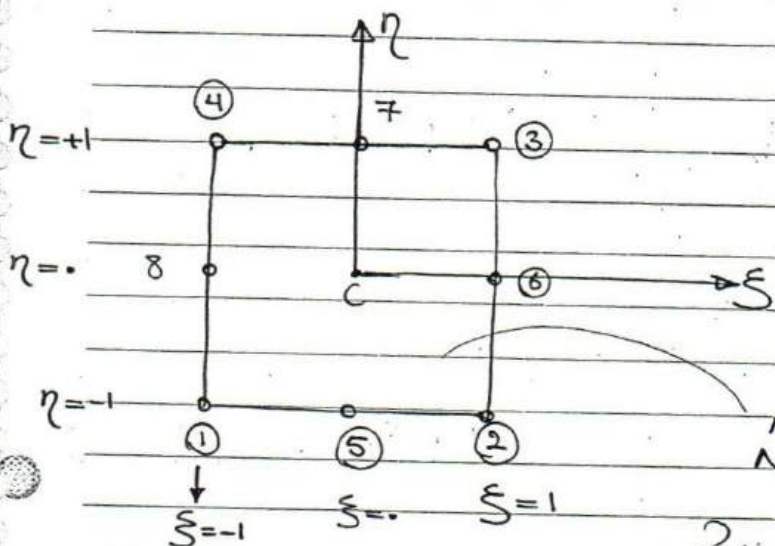
$$P(x, y) = \langle 1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^2y \quad xy^2 \quad x^2y^2 \rangle$$



Serendipity

المان های

المان های Serendipity. المان هایی هستند که لااقل از آنجا ظاهر می شوند

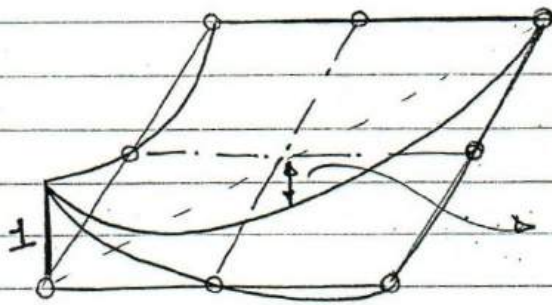


$$-1 \leq \xi = \frac{x - x_c}{a} \leq 1$$

$$-1 \leq \eta = \frac{y - y_c}{b} \leq 1$$

$N_i(\xi, \eta)$

ضریب شکل المان های Serendipity را با N_i نمایش می دهیم

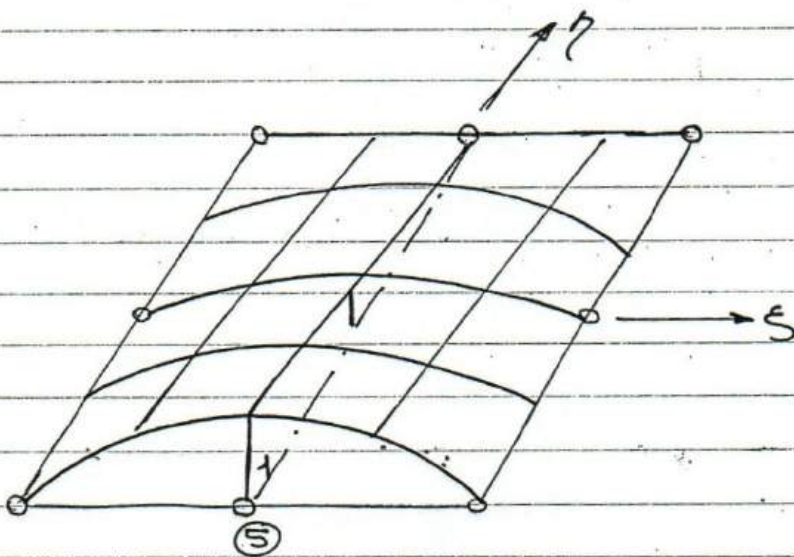


صفر نباشد

$$\hat{N}_5(\xi, \eta) = N_5(\xi, \eta) = \frac{1}{2}(1-\xi)(1-\eta)(1+\xi)$$

$$\rightarrow \hat{N}_5 = \frac{1}{2}(1-\xi)(1+\xi)(1-\eta)$$

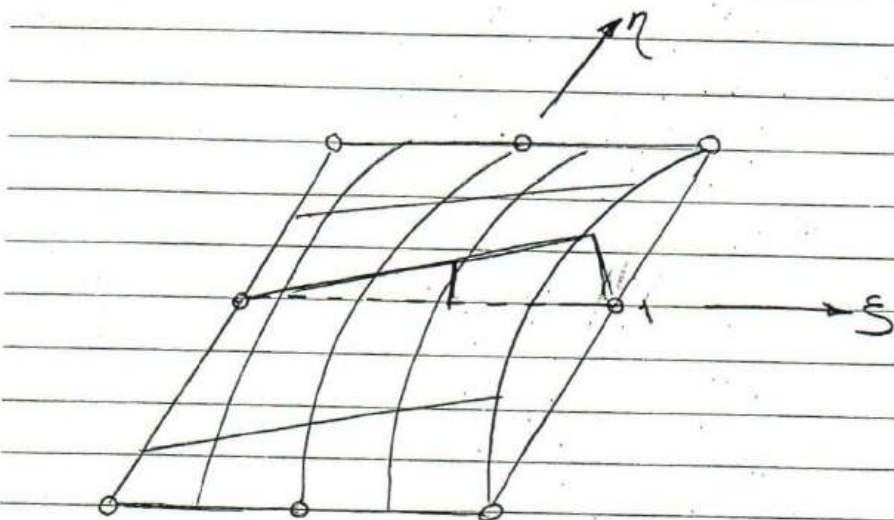
ترک های η^2 ندارد
ترک های $\xi^2 \eta^2$ ندارد



$$\hat{N}_6(\xi, \eta) = \frac{1}{2}(1-\eta)(1+\xi)(1+\eta)$$

$$\hat{N}_6(\xi, \eta) = \frac{1}{2}(1-\eta)(1+\eta)(1+\xi)$$

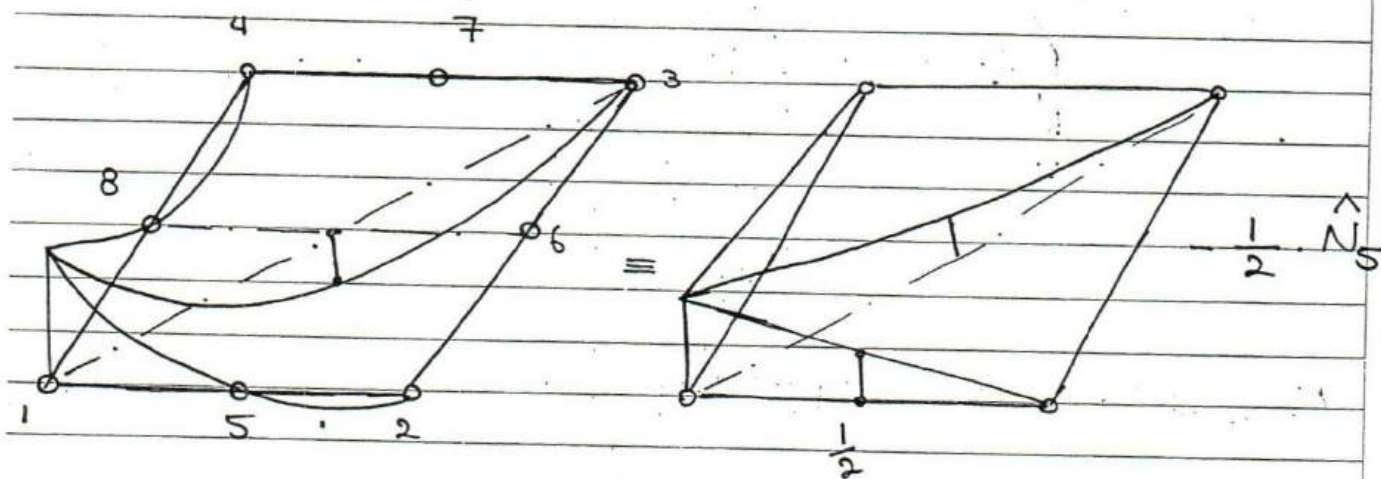
ترک های $\xi^2 \eta^2$ ندارد

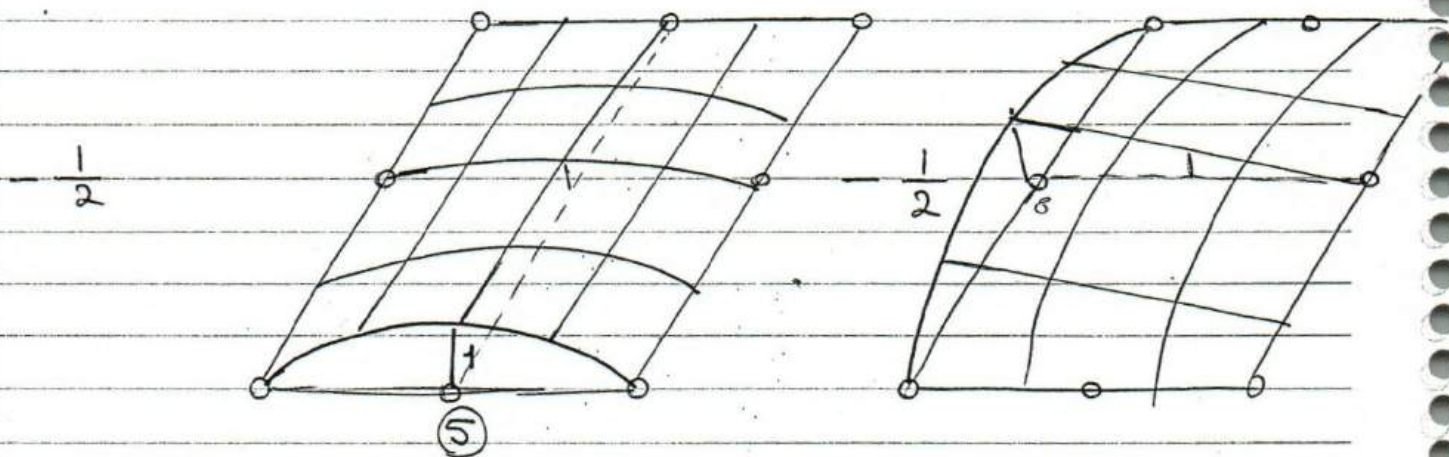
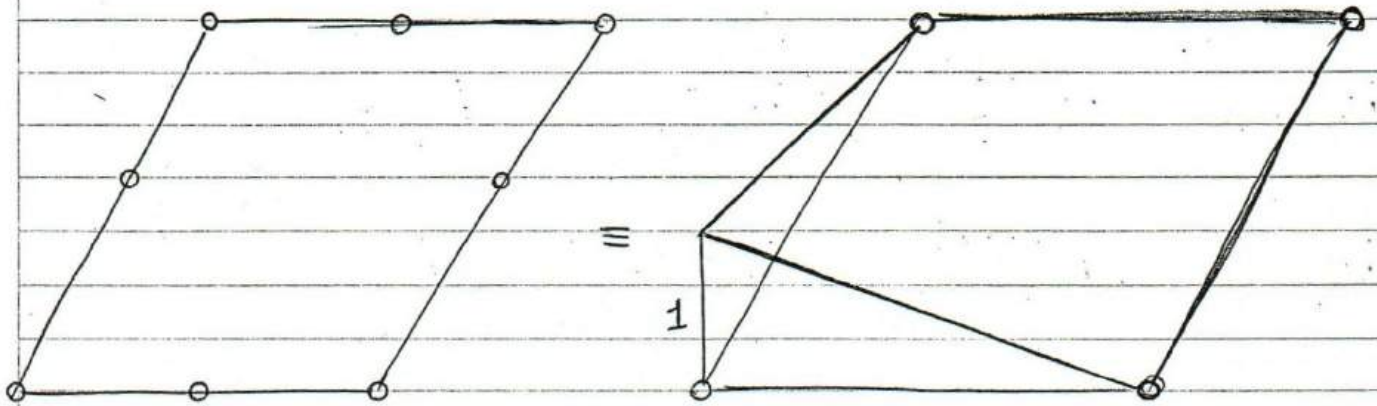


$$\hat{N}_7(\xi, \eta) = \frac{1}{2}(1 - \xi^2)(1 + \eta)$$

$$\hat{N}_8(\xi, \eta) = \frac{1}{2}(1 - \xi)(1 - \eta^2)$$

حال باید ضریب شکل گره‌های نوشته و اندیس بیاریم





$$\hat{N}_1(\xi, \eta) = \overset{4\text{-noded}}{N}_1(\xi, \eta) - \frac{1}{2} \hat{N}_5(\xi, \eta) - \frac{1}{2} \hat{N}_8(\xi, \eta)$$

$$\hat{N}_1(\xi, \eta) = \frac{1}{4} (1-\xi)(1-\eta) - \frac{1}{4} (1-\xi^2)(1-\eta) - \frac{1}{4} (1-\xi)(1-\eta^2)$$

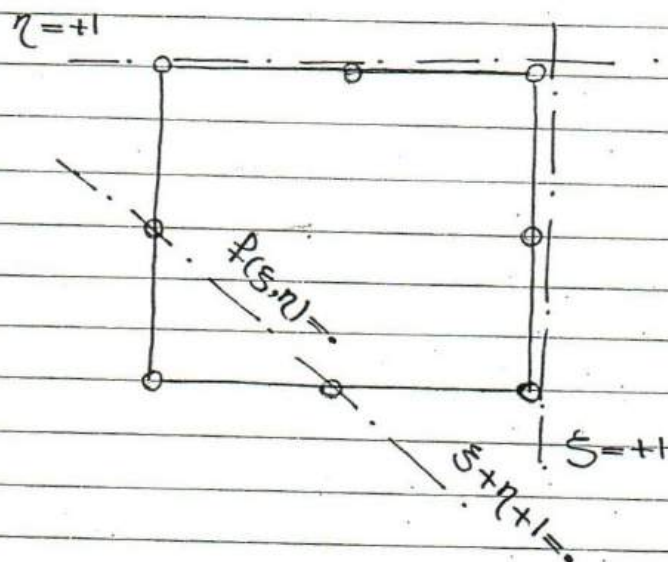
$$\hat{N}_2(\xi, \eta) = \overset{4\text{-noded}}{N}_2(\xi, \eta) - \frac{1}{2} \hat{N}_5(\xi, \eta) - \frac{1}{2} \hat{N}_6(\xi, \eta)$$

$$\hat{N}_2(\xi, \eta) = N_2(\xi, \eta) - \frac{1}{2} \hat{N}_5(\xi, \eta) - \frac{1}{2} \hat{N}_6(\xi, \eta)$$

$$\hat{N}_3(\xi, \eta) = N_3(\xi, \eta) - \frac{1}{2} \hat{N}_6(\xi, \eta) - \frac{1}{2} \hat{N}_7(\xi, \eta)$$

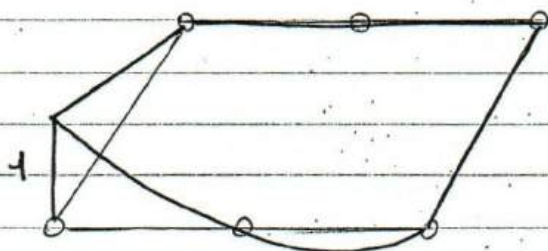
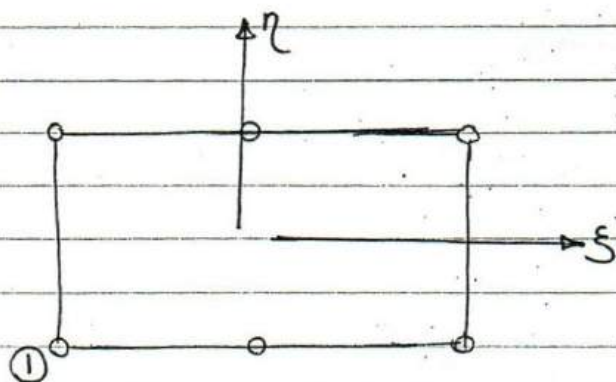
$$\hat{N}_4(\xi, \eta) = N_4(\xi, \eta) - \frac{1}{2} \hat{N}_7(\xi, \eta) - \frac{1}{2} \hat{N}_8(\xi, \eta)$$

روش دایر:

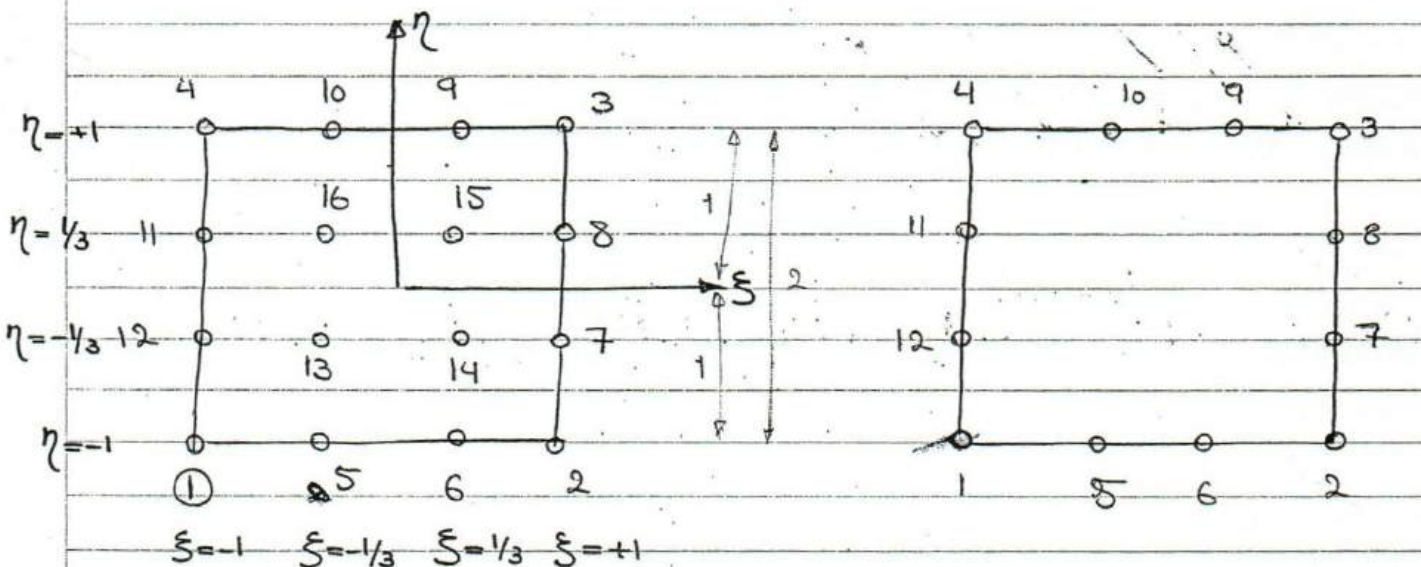


$$\hat{N}_1(\xi, \eta) = -\frac{1}{4} (1 - \xi)(1 - \eta)(1 + \xi + \eta)$$

شرط استفاده از این روش این است که خطوط خارج از المان هم دیده افتد.

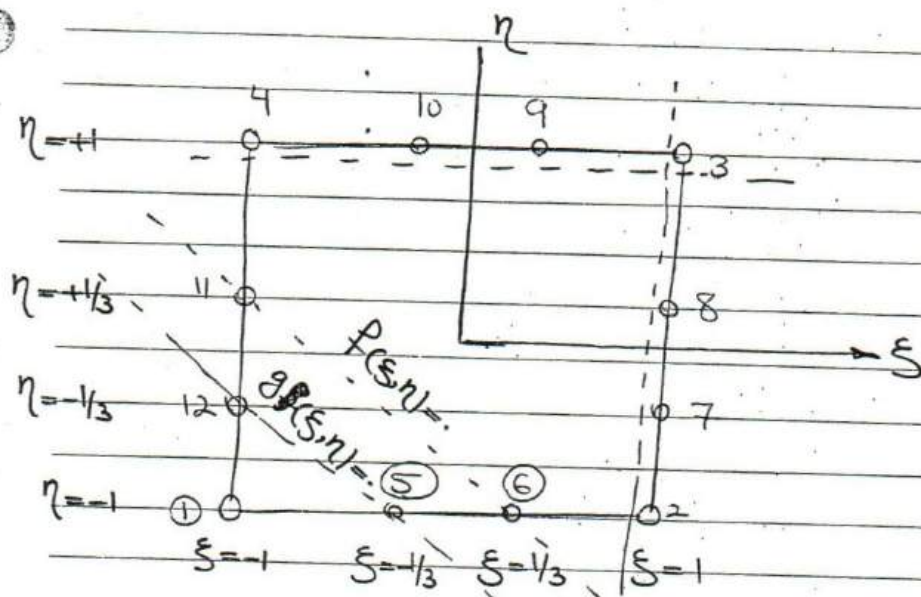


$$N_1 = -\frac{1}{4} \xi(1-\xi)(1-\eta)$$

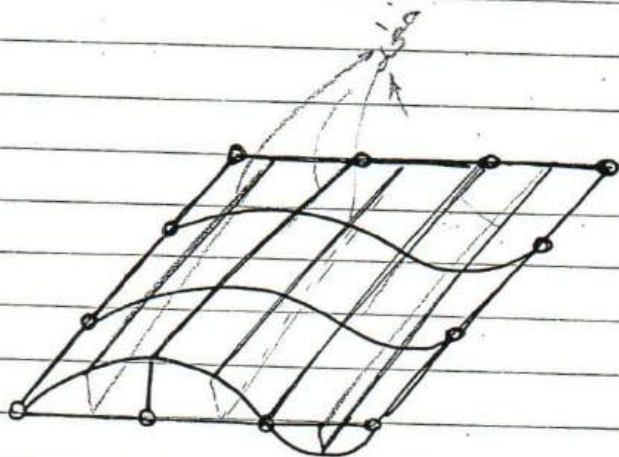


$$N_1(\xi, \eta) = \frac{81}{256} \left(\frac{1}{9} - \xi^2\right)(1-\xi)\left(\frac{1}{9} - \eta^2\right)(1-\eta)$$

$$N_5(\xi, \eta) = \frac{-243}{256} \left(\frac{1}{3} - \xi\right)(1-\xi^2)\left(\frac{1}{9} - \eta^2\right)(1-\eta)$$



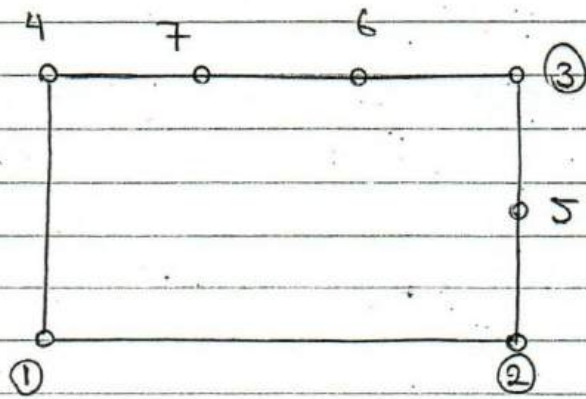
$$\hat{N}_5(\xi, \eta) = \frac{27}{32} (1 - \xi^2) \left(\frac{1}{3} - \xi\right) (1 - \eta)$$



$$\hat{N}_1(\xi, \eta) = N_1 \quad \text{4-noded} \quad \frac{2}{3} \hat{N}_5 - \frac{1}{3} \hat{N}_6 - \frac{2}{3} \hat{N}_{12} - \frac{1}{3} \hat{N}_{11}$$

از روش دویم:

$$\hat{N}_1(\xi, \eta) = \frac{1}{8} (1 - \xi)(1 - \eta) f(\xi, \eta) g(\xi, \eta)$$



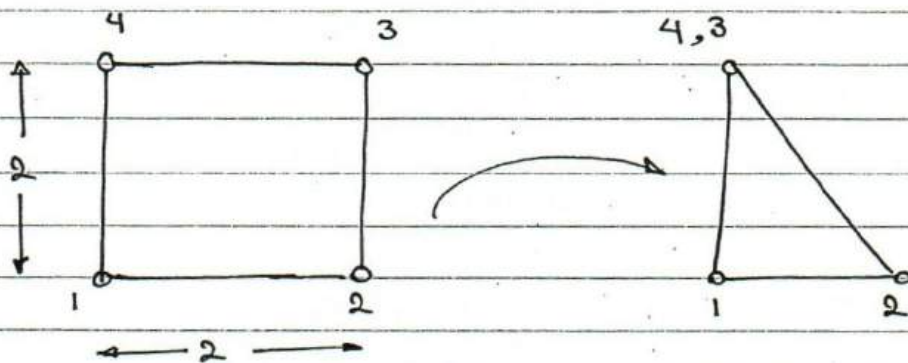
$$\hat{N}_1(\xi, \eta) = N_1 \quad \text{4-noded}$$

$$\hat{N}_2 = N_2(\xi, \eta) - \frac{1}{2} \hat{N}_5(\xi, \eta) \quad \text{4-noded}$$

$$\hat{N}_3 = N_3 - \frac{1}{2} \hat{N}_5(\xi, \eta) - \frac{2}{3} \hat{N}_6(\xi, \eta) - \frac{1}{3} \hat{N}_7(\xi, \eta) \quad \text{4-node}$$

$$\hat{N}_4 = N_4 - \frac{2}{3} \hat{N}_7(\xi, \eta) - \frac{1}{3} \hat{N}_6(\xi, \eta) \quad \text{4-noded}$$

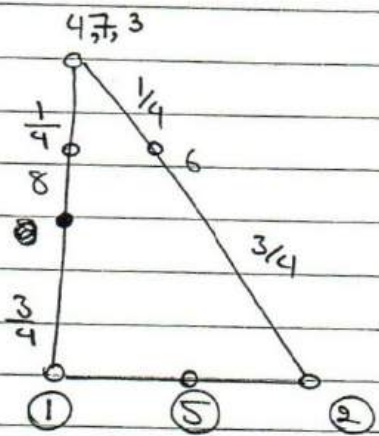
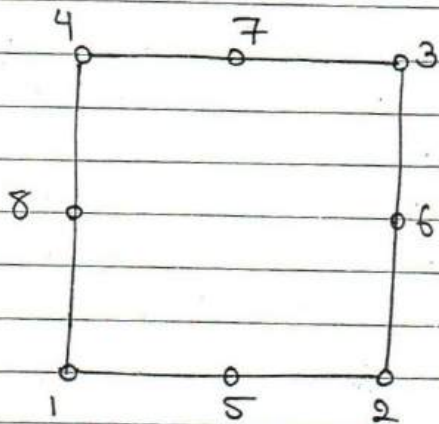
تقرین : چپ لستہ اندر درجہ دار حرکت العان کے درجہ بہ ہم وصل کسٹم تراف سے حذف می شود۔



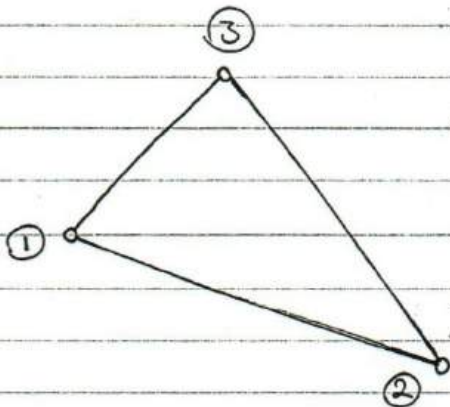
$$\langle 1 \times y \quad xy \rangle$$

$$\langle 1 \times y \rangle$$

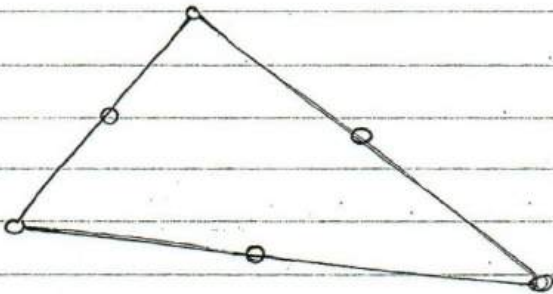
Subject Date



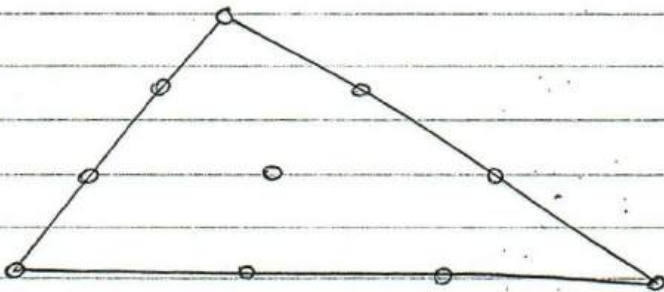
المان های مثلث مرتبه بالا ← با معرفی دستگاه مختصات مثلث



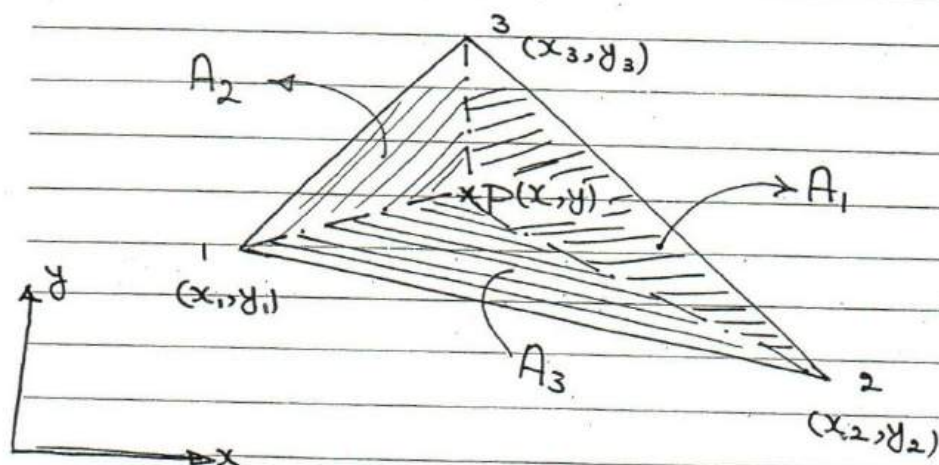
$$\Rightarrow N_i = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$



$$\Rightarrow O(2) \quad 6\text{-Noded.}$$

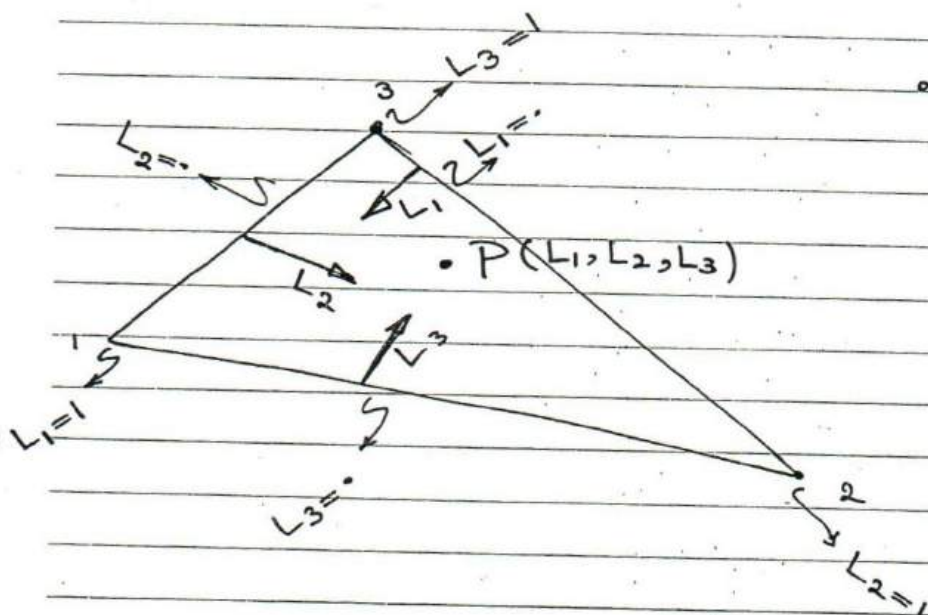


$$O(3) \quad 10\text{-Noded.}$$



Triangular (area) Coordinate

دستگاه مختصات مثلثی (مساحت)



$$0 \leq L_1, L_2, L_3 \leq 1$$

$$L_1 + L_2 + L_3 = 1$$

رابطه بین مختصات بارتری و مختصات مثلثی

$$x = x_1 L_1 + x_2 L_2 + x_3 L_3$$

1 " $\rightarrow L_1 = 1 \rightarrow L_2 = L_3 = 0 \Rightarrow x_1 = x_1 \checkmark$

2 " $\rightarrow$

3 " $\rightarrow$

$$y = y_1 L_1 + y_2 L_2 + y_3 L_3$$

$$\begin{cases} 1 = L_1 + L_2 + L_3 \\ x = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 \\ y = \beta_1 L_1 + \beta_2 L_2 + \beta_3 L_3 \end{cases} \Rightarrow \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}$$

① نوشته ② نوشته ③ نوشته

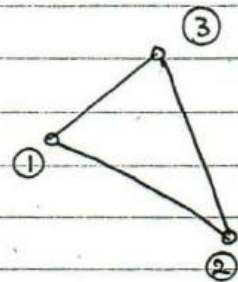
برای استفاده ماتریس C الی مشتق است

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$L_1 = \frac{1}{2A} (\alpha_1 + \beta_1 x + \gamma_1 y) \equiv N_1 \quad \text{3-Noded}$$

$$L_2 = \frac{1}{2A} (\alpha_2 + \beta_2 x + \gamma_2 y) \equiv N_2 \quad \text{3-Noded}$$

$$L_3 = \frac{1}{2A} (\alpha_3 + \beta_3 x + \gamma_3 y) \equiv N_3 \quad \text{3-Noded}$$



$$L_1 = \frac{1}{2A} \left\{ 1 \cdot \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ y_2 & y_3 \end{vmatrix} x + \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} y \right\}$$

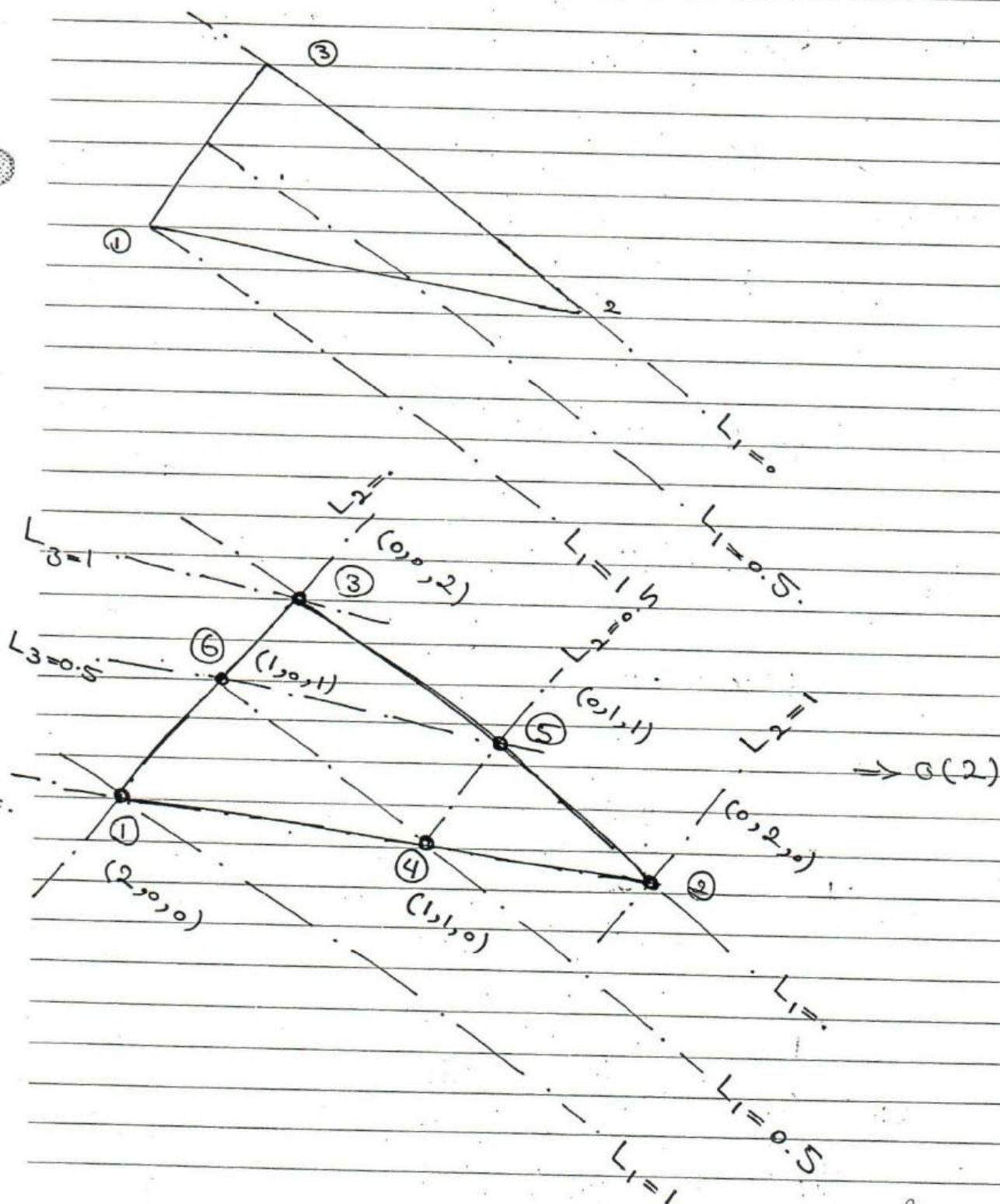
$$= \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{2A_1}{2A} = \frac{A_1}{A}$$

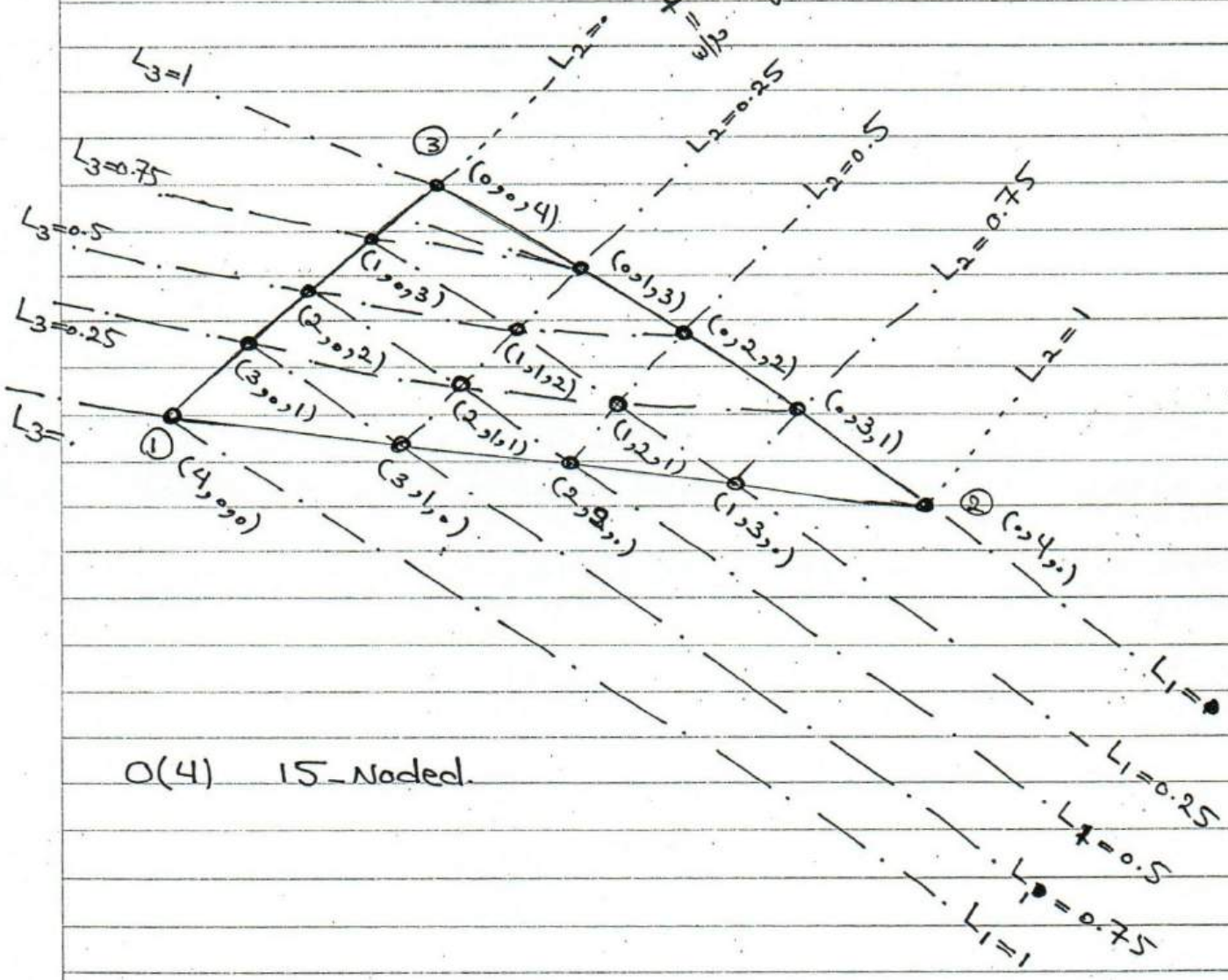
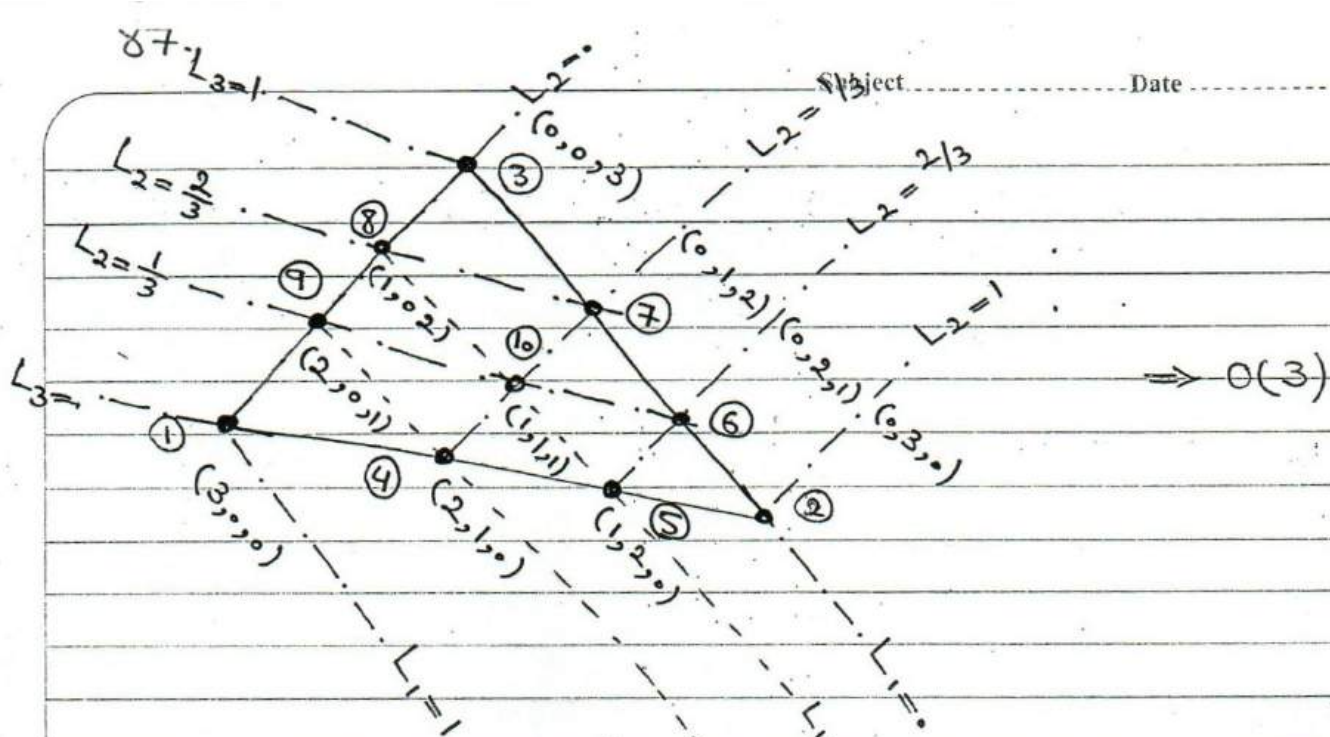
(P) ② ③

$$L_2 = \frac{1}{2A} \quad 2A_2 = \frac{A_2}{A}$$

$$\Rightarrow L_1 + L_2 + L_3 = 1.$$

$$L_3 = \frac{2A_3}{2A} = \frac{A_3}{A}$$





با استفاده از روش لایبرانت در محققات مسافتی، نتایج شکل به صورت زیر تعریف می شود:

$$N_{p,q,r}(L_1, L_2, L_3) = N_p(L_1) N_q(L_2) N_r(L_3)$$

$$N_p(L_1) = \begin{cases} \prod_{i=1}^p \left(\frac{m L_1 - i + 1}{i} \right) & p \geq 1 \\ 1 & p = 0 \end{cases}$$

رسته المان: $m \Rightarrow m = p + q + r$
order of element

مثال برای المان مرتبه دوم:

$$N_1 \equiv N_{200}(L_1, L_2, L_3) = N_2(L_1) N_0(L_2) N_0(L_3) = L_1(2L_1 - 1)$$

$$N_2 \equiv N_{020}(L_1, L_2, L_3) = N_0(L_1) N_2(L_2) N_0(L_3) = L_2(2L_2 - 1)$$

$$N_3 \equiv N_{002}(L_1, L_2, L_3) = N_0(L_1) N_0(L_2) N_2(L_3) = L_3(2L_3 - 1)$$

$$N_4 \equiv N_{110}(L_1, L_2, L_3) = N_1(L_1) N_1(L_2) N_0(L_3) = (2L_1)(2L_2) = 4L_1 L_2$$

$$N_5 \equiv N_{011}(L_1, L_2, L_3) = N_0(L_1) N_1(L_2) N_1(L_3) = 4L_2 L_3$$

$$N_6 \equiv N_{101}(L_1, L_2, L_3) = N_1(L_1) N_0(L_2) N_1(L_3) = 4L_1 L_3$$

$$O(2) \Rightarrow m=2$$

$$N_0(L_1) = 1$$

$$N_1(L_1) = \prod_{i=1}^1 \left(\frac{2L_1 - i + 1}{i} \right) = \frac{2L_1 - 1 + 1}{1} = 2L_1$$

$$N_2(L_1) = \prod_{i=1}^2 \left(\frac{2L_1 - i + 1}{i} \right) = \left(\frac{2L_1 - 1 + 1}{1} \right) \left(\frac{2L_1 - 2 + 1}{2} \right)$$

$$= L_1(2L_1 - 1)$$

$$N_0(L_2) = 1$$

$$N_0(L_3) = 1$$

$$N_1(L_2) = 2L_2$$

$$N_1(L_3) = 2L_3$$

$$N_2(L_2) = L_2(2L_2 - 1)$$

$$N_2(L_3) = L_3(2L_3 - 1)$$

$$N_1 = L_1(2L_1 - 1) \Rightarrow L_1 = N_1 = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$

Subject Date

$$O(2) \Rightarrow m=2$$

$$N_0(L_1) = 1$$

$$N_1(L_1) = \prod_{i=1}^1 \left(\frac{2L_1 - i + 1}{i} \right) = \frac{2L_1 - 1 + 1}{1} = 2L_1$$

$$N_2(L_1) = \prod_{i=1}^2 \left(\frac{2L_1 - i + 1}{i} \right) = \left(\frac{2L_1 - 1 + 1}{1} \right) \left(\frac{2L_1 - 2 + 1}{2} \right) \\ = L_1(2L_1 - 1)$$

$$N_0(L_2) = 1$$

$$N_0(L_3) = 1$$

$$N_1(L_2) = 2L_2$$

$$N_1(L_3) = 2L_3$$

$$N_2(L_2) = L_2(2L_2 - 1)$$

$$N_2(L_3) = L_3(2L_3 - 1)$$

$$N_1 = L_1(2L_1 - 1) \Rightarrow L_1 = N_1 = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$

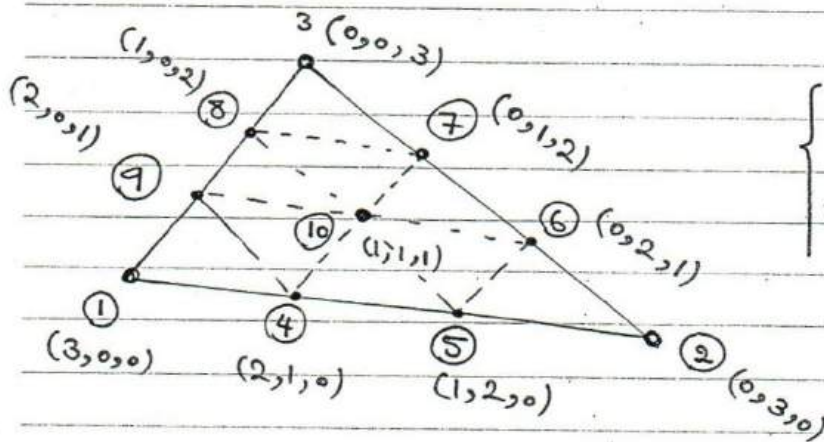
$$N_P(L_j) = \begin{cases} \prod_{i=1}^P \left(\frac{m L_j - i + 1}{i} \right) & P \geq 1 \\ 1 & P = 0 \end{cases}$$

تعیین قواعد شکل بر اساس درجته در مفصلیات مساحی
درش

$$N_{Pqr}(L_1, L_2, L_3) = N_P(L_1) N_Q(L_2) N_R(L_3)$$

$$N_P(L_1) = \begin{cases} \prod_{i=1}^P \left(\frac{mL_1 - i + 1}{i} \right) & P \geq 1 \\ 1 & P = 0 \end{cases}$$

$$\Rightarrow m = P + Q + R$$



$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \sum_{i=1}^{N_{\text{OND}}} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$N_0(L_1) = 1$$

$$N_1(L_1) = \frac{3L_1 - 1 + 1}{1} = 3L_1$$

$$N_2(L_1) = \left(\frac{3L_1 - 1 + 1}{1} \right) \left(\frac{3L_1 - 2 + 1}{2} \right) = \frac{3}{2} L_1 (3L_1 - 1)$$

$$N_3(L_1) = \left(\frac{3L_1 - 1 + 1}{1} \right) \left(\frac{3L_1 - 2 + 1}{2} \right) \left(\frac{3L_1 - 3 + 1}{3} \right) = \frac{1}{2} L_1 (3L_1 - 1) (3L_1 - 2)$$

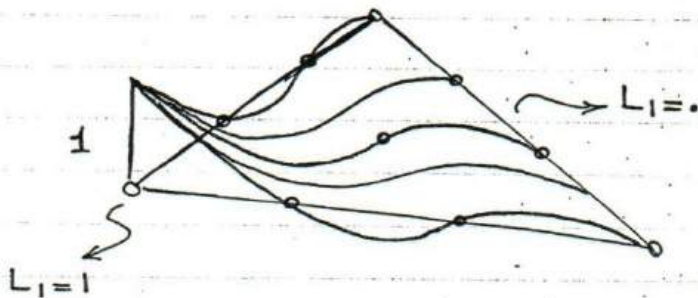
$$\rightarrow N_0(L_1) = 1$$

$$\rightarrow N_1(L_1) = 3L_1$$

$$\rightarrow N_2(L_1) = \alpha L_1$$

$$\rightarrow N_3(L_1) = \beta L_1^2$$

$$N_1 = N_{300}(L_1, L_2, L_3) = \frac{1}{2} \cdot L_1 (3L_1 - 1)(3L_1 - 2)$$



$$N_4 = \frac{3}{2} L_1 (3L_1 - 1)(3L_2)$$

$$N_{10} = 27 L_1 \cdot L_2 \cdot L_3$$

ماتریس B :

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \sum_{i=1}^{n_{\text{ND}}} \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$(x, y) \longleftrightarrow (L_1, L_2, L_3)$$

$$\xrightarrow{\frac{1}{2A} \beta_1}$$

$$\xrightarrow{\frac{1}{2A} \beta_2}$$

$$\xrightarrow{\frac{1}{2A} \beta_3}$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial L_1} \frac{\partial L_1}{\partial x} + \frac{\partial N_i}{\partial L_2} \frac{\partial L_2}{\partial x} + \frac{\partial N_i}{\partial L_3} \frac{\partial L_3}{\partial x} \Rightarrow i = 1, 2, \dots, 10$$

3-Noded

$$\rightarrow L_i \equiv N_i = \frac{1}{2A} (\alpha_i + \beta_i x + \gamma_i y)$$

Subject Date

$$\rightarrow \frac{\delta N_i}{\delta x} = \frac{1}{2A} \sum_{K=1}^3 \frac{\delta N_i}{\delta L_K} \cdot \beta_K \quad i=1,2,\dots,10$$

$$\frac{1}{2} A \delta_1$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial L_1} \left[\frac{\partial L_1}{\partial y} \right] + \frac{\partial N_i}{\partial L_2} \frac{\partial L_2}{\partial y} + \frac{\partial N_i}{\partial L_3} \frac{\partial L_3}{\partial y}$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{2A} \sum_{K=1}^3 \frac{\partial N_i}{\partial L_K} \cdot \delta_K$$

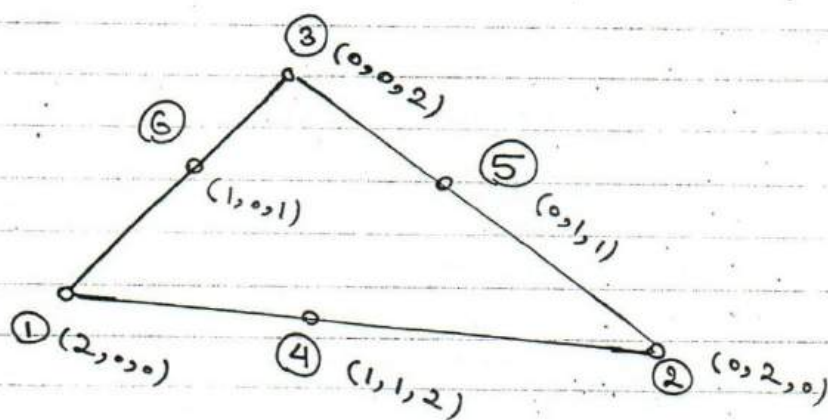
$$i = 1, 2, \dots, 10$$

نمونه Node ها

$$B_i = \frac{1}{2A} \begin{bmatrix} \sum_{K=1}^3 \frac{\partial N_i}{\partial L_K} \cdot \bar{P}_K & \sum_{K=1}^3 \frac{\partial N_i}{\partial L_K} \delta_K \\ \sum_{K=1}^3 \frac{\partial N_i}{\partial L_K} \delta_K & \sum_{K=1}^3 \frac{\partial N_i}{\partial L_K} \bar{P}_K \end{bmatrix}$$

(L_1, L_2, L_3)
 $i = 1, 2, \dots, \text{NODE}$

به عنوان مثال برای یک المان مرتبه دوم 6 گرهی:



$$N_0(L_1) = 1$$

$$N_1(L_1) = 2L_1$$

$$N_2(L_1) = 2L_1 \left(\frac{2L_1 - 2 + 1}{2} \right) = L_1(2L_1 - 1)$$

i	N_i	$\frac{\partial N_i}{\partial L_1}$	$\frac{\partial N_i}{\partial L_2}$	$\frac{\partial N_i}{\partial L_3}$
①	$L_1(2L_1-1)$	$4L_1-1$	.	.
②	$L_2(2L_2-1)$	.	$4L_2-1$	.
③	$L_3(2L_3-1)$	.	.	$4L_3-1$
④	$4L_1 \cdot L_2$	$4L_2$	$4L_1$	.
⑤	$4L_2 \cdot L_3$	.	$4L_3$	$4L_2$
⑥	$4L_1 \cdot L_3$	$4L_3$	.	$4L_1$

$$\rightarrow \underset{(2 \times 2)}{K_{ij}} = \int \underset{\substack{\text{---} \overset{T}{\text{---}} (3 \times 3) \\ \underset{(L_1, L_2, L_3)}{\text{---}} (2 \times 3)}}{\underset{\text{---}}{B_i} \cdot \underset{\text{---}}{D} \cdot \underset{\substack{\text{---} \text{---} (3 \times 2) \\ \underset{(L_1, L_2, L_3)}{\text{---}}}}{\underset{\text{---}}{B_j}} \cdot t \cdot dA \quad \Rightarrow i, j = 1, 2, \dots, \text{NOD}}$$

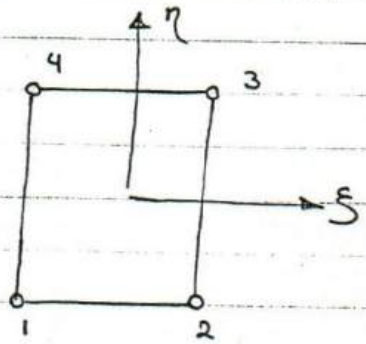
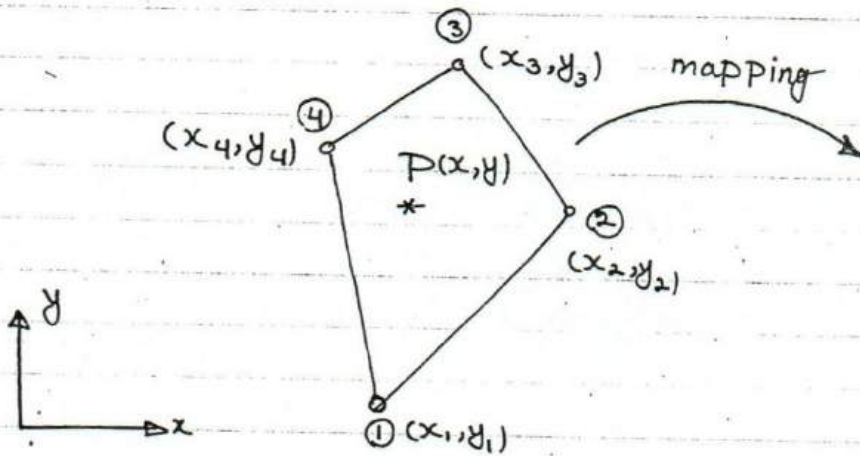
این اشتغال به روش عددی باید محاسبه شود [روش گوس و ...]

← برای محاسبه به روش دستی داریم:

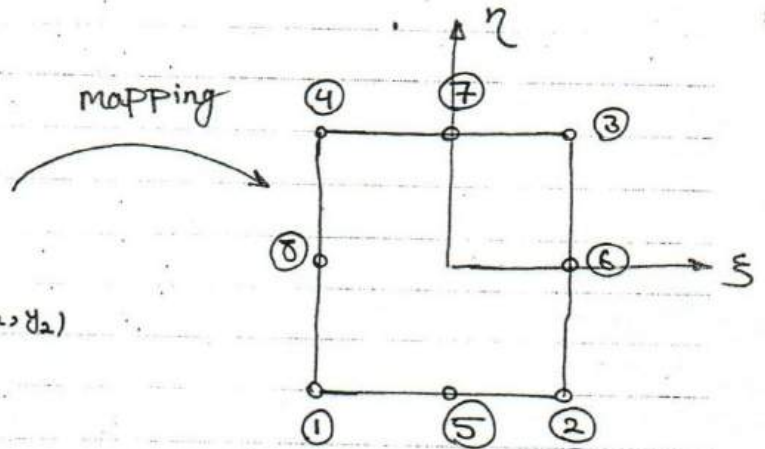
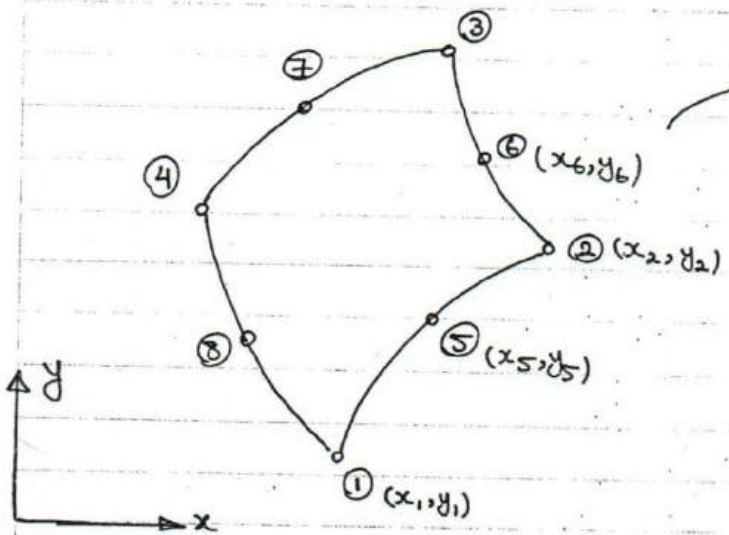
$$\int_A \overset{a}{L_1} \cdot \overset{b}{L_2} \cdot \overset{c}{L_3} \cdot dA = \frac{a! \cdot b! \cdot c!}{(a+b+c+2)!} \cdot (2A)$$

Iso-parametric element.

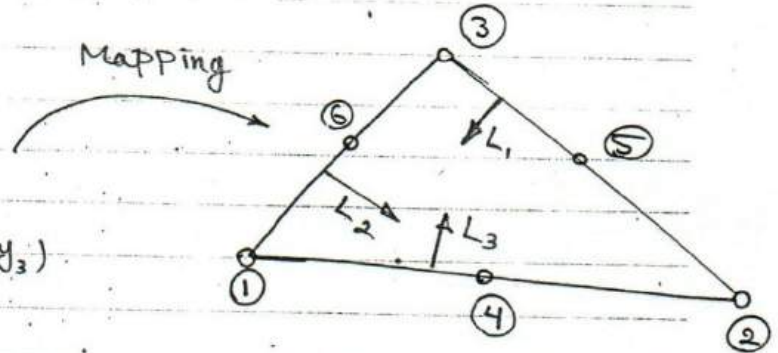
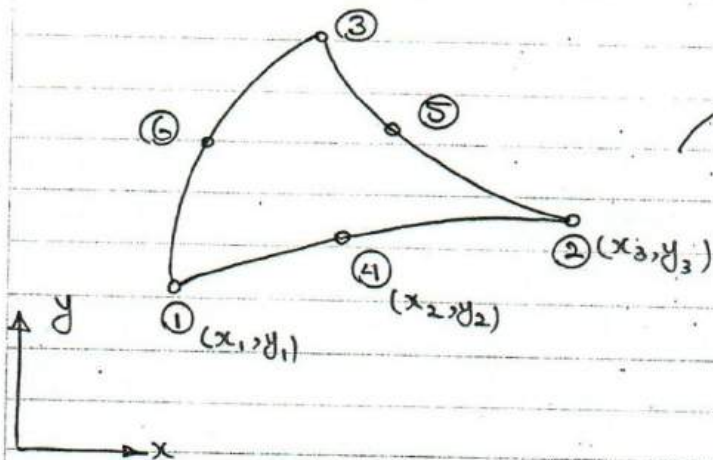
المان های ایسوپارامتریک:



$$N_i(\xi, \eta) = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta)$$



$$N_i(\xi, \eta) =$$

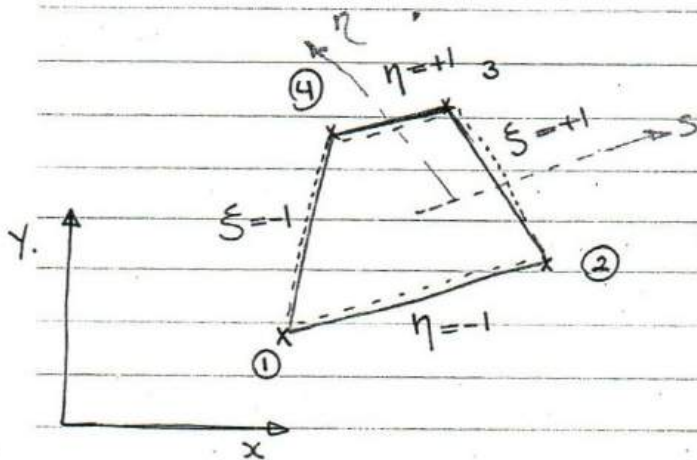


$$N_i(L_1, L_2, L_3)$$

به عنوان مثال برای المان اینترپولاسیون ۴ نره ای

$$x = x_1 N_1 + x_2 N_2 + x_3 N_3 + x_4 N_4 = \sum_{i=1}^4 x_i N_i(\xi, \eta)$$

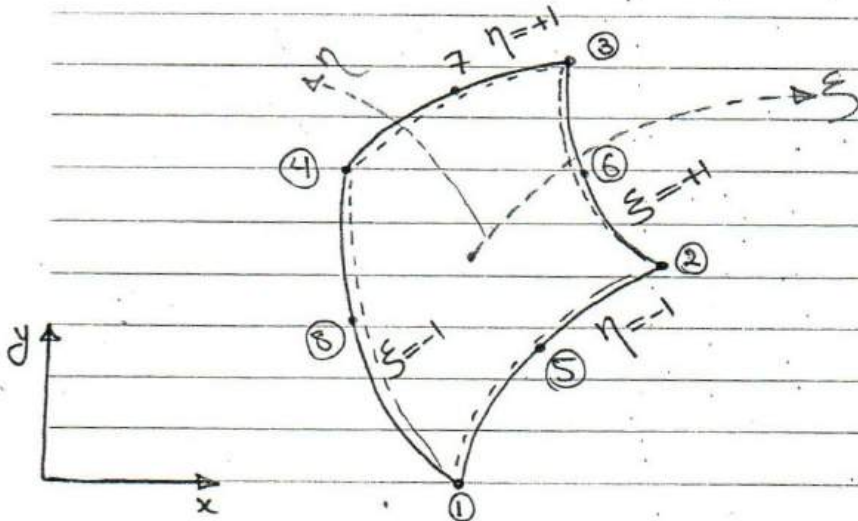
$$y = y_1 N_1 + y_2 N_2 + y_3 N_3 + y_4 N_4 = \sum_{i=1}^4 y_i N_i(\xi, \eta)$$



برای المان ۸ نره ای:

$$x = x_1 N_1 + x_2 N_2 + \dots + x_8 N_8 = \sum_{i=1}^8 x_i N_i(\xi, \eta)$$

$$y = y_1 N_1 + y_2 N_2 + \dots + y_8 N_8 = \sum_{i=1}^8 y_i N_i(\xi, \eta)$$



$$x = \sum_{i=1}^{nonD} x_i \cdot N_i(\xi, \eta)$$

$$y = \sum_{i=1}^{nonD} y_i \cdot N_i(\xi, \eta)$$

$$\rightarrow B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \cdot \\ \cdot & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\begin{cases} \frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \\ \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \end{cases} *$$

$$\Rightarrow \begin{cases} \frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \eta} \end{cases}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

← مائٹریس ژا کوپس

Jacobian matrix

$$\bar{J} = \begin{bmatrix} \sum_{i=1}^{nond} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{nond} y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{nond} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{nond} y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \delta \eta \\ \delta y \end{matrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \bar{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

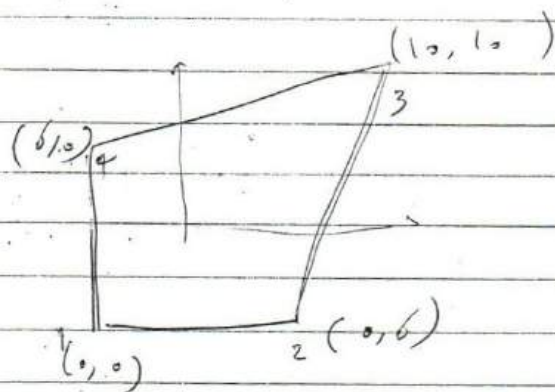
$$\frac{\partial \nu}{\partial \xi} = \frac{\partial \nu}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \nu}{\partial y} \frac{\partial y}{\partial \xi}$$

$$N_1 = (1-\xi)(1+\eta)$$

$$N_2 = (1+\xi)(1+\eta)$$

$$N_3 = (1+\xi)(1-\eta)$$

$$N_4 = (1-\xi)(1-\eta)$$

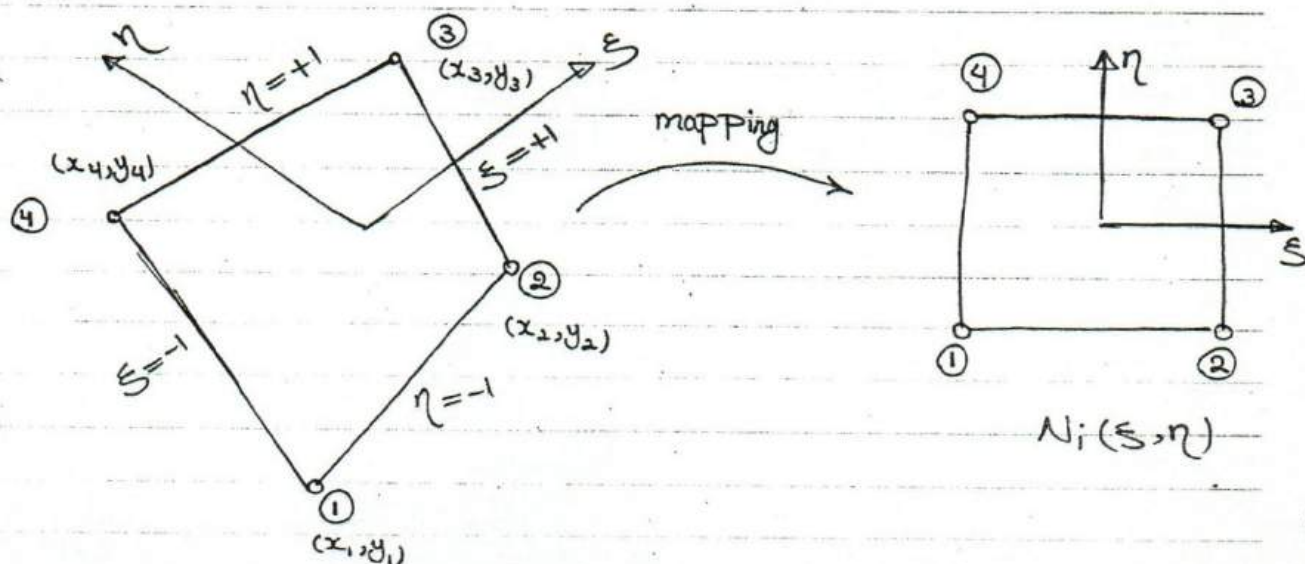


$$x = N_1 \eta_1$$

$$-(1+\eta) = \frac{\partial \nu}{\partial \eta} \cdot (\eta_1(\eta+1)) + \frac{\partial \nu}{\partial \xi} \cdot (\eta_1(\eta+1))$$

$$(1-\xi) = \frac{\partial \nu}{\partial \eta} (\eta_1(1-\xi)) + \frac{\partial \nu}{\partial \xi} (\eta_1(1-\xi))$$

المان های اینر و پراستریک :



$$x = \sum_{(\xi, \eta)} x_i N_i(\xi, \eta)$$

$$y = \sum_{(\xi, \eta)} y_i N_i(\xi, \eta)$$

$$\tilde{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \tilde{J} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\tilde{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\Rightarrow \tilde{J} = \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix} = \begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \xi} & \sum y_i \frac{\partial N_i}{\partial \xi} \\ \sum x_i \frac{\partial N_i}{\partial \eta} & \sum y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \dot{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \frac{1}{|\dot{J}|} \begin{bmatrix} \dot{J}_{22} & -\dot{J}_{12} \\ -\dot{J}_{21} & \dot{J}_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$= \frac{1}{|\dot{J}|} \begin{Bmatrix} \dot{J}_{22} \cdot \frac{\partial N_i}{\partial \xi} - \dot{J}_{12} \cdot \frac{\partial N_i}{\partial \eta} \\ -\dot{J}_{21} \cdot \frac{\partial N_i}{\partial \xi} + \dot{J}_{11} \cdot \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

به عنوان مثال برای یک المان 4 گرهی این روابط مرتب می شود:

$$N_i(\xi, \eta) = \frac{1}{4} (1 + \xi_i \xi) (1 + \eta_i \eta)$$

$$\frac{\partial N_i}{\partial \xi} = \frac{1}{4} \xi_i (1 + \eta_i \eta)$$

$$\frac{\partial N_i}{\partial \eta} = \frac{1}{4} \eta_i (1 + \xi_i \xi)$$

$$\dot{J}_{11} = \sum_{K=1}^4 x_K \cdot \frac{1}{4} \xi_K (1 + \eta_K \eta)$$

$$\dot{J}_{12} = \sum_{K=1}^4 y_K \cdot \frac{1}{4} \xi_K (1 + \eta_K \eta)$$

$$\dot{J}_{21} = \sum_{K=1}^4 x_K \cdot \frac{1}{4} \eta_K (1 + \xi_K \xi)$$

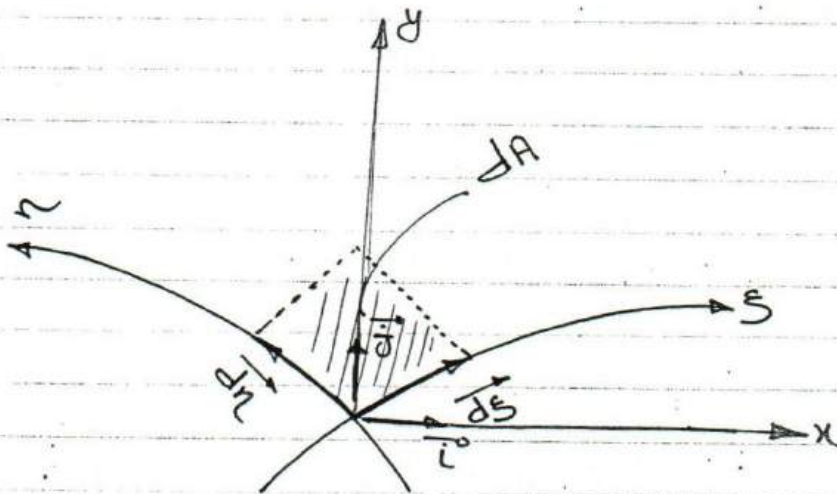
$$\dot{J}_{22} = \sum_{K=1}^4 y_K \cdot \frac{1}{4} \eta_K (1 + \xi_K \xi)$$

$$|\dot{\mathbf{j}}|_{(\xi, \eta)} = \dot{J}_{11} \cdot \dot{J}_{22} - \dot{J}_{12} \cdot \dot{J}_{21}$$

$$\frac{\partial N_i}{\partial x}_{(\xi, \eta)} = \frac{1}{|\dot{\mathbf{j}}|} \left\{ \dot{J}_{22}_{(\xi)} \cdot \frac{1}{4} \xi_i (1 + \eta_i \eta) - \dot{J}_{12}_{(\eta)} \cdot \frac{1}{4} \eta_i (1 + \xi_i \xi) \right\}$$

$$\frac{\partial N_i}{\partial y}_{(\xi, \eta)} = \frac{1}{|\dot{\mathbf{j}}|} \left\{ -\dot{J}_{21}_{(\xi)} \cdot \frac{1}{4} \xi_i (1 + \eta_i \eta) + \dot{J}_{11}_{(\eta)} \cdot \frac{1}{4} \eta_i (1 + \xi_i \xi) \right\}$$

$$\rightarrow K_{ij} = \int_A \tilde{B}_i^T \cdot \underline{D} \cdot B_j \cdot t \cdot dA$$



$$dA = |\vec{d\xi} \times \vec{d\eta}|$$



$$d\vec{\xi} = \frac{\partial x}{\partial \xi} \cdot d\xi \cdot \vec{i} + \frac{\partial y}{\partial \xi} \cdot d\xi \cdot \vec{j}$$

$$d\vec{\eta} = \frac{\partial x}{\partial \eta} \cdot d\eta \cdot \vec{i} + \frac{\partial y}{\partial \eta} \cdot d\eta \cdot \vec{j}$$

$$dA = |d\vec{\xi} \times d\vec{\eta}| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} \cdot d\xi \cdot d\eta$$

$$\rightarrow dA = |\vec{j}| \cdot d\xi \cdot d\eta$$

$$\rightarrow K_{ij} = \int_{-1}^{+1} \int_{-1}^{+1} \left[\tilde{B}_i^T(\xi, \eta) \cdot D \cdot \tilde{B}_j(\xi, \eta) + |\vec{j}| \cdot d\xi \cdot d\eta \right] \cdot f(\xi, \eta)$$

$$\equiv \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta$$

روش انتگرال گیری عددی :

تعداد نقاط انتگرال گیری

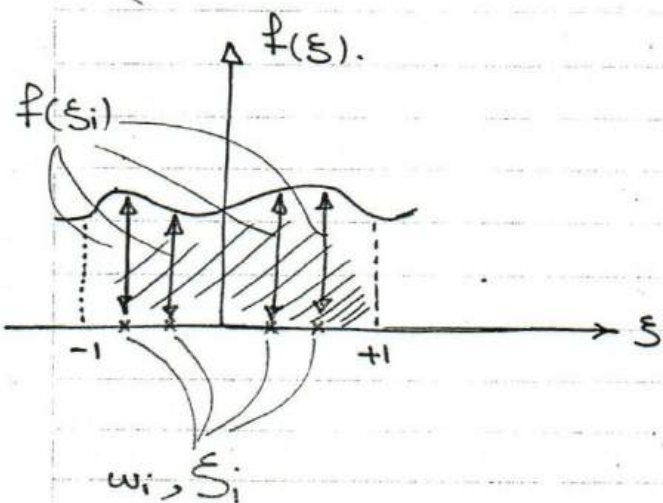
موقعیت نقاط انتگرال گیری

$$\Rightarrow \int_{-1}^{+1} f(\xi) \cdot d\xi = \sum_{i=1}^3 f(\xi_i) \cdot w_i$$

برای حالت یک بعدی

مقدار وزن در هر نقطه انتگرال گیری

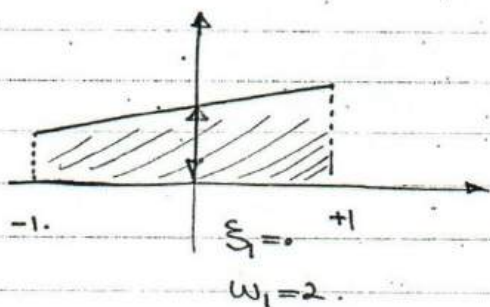
نقطه انتگرال گیری



Gauss. integration method. روش انتگرال گیری گوس :

بر اساس این روش برای m نقطه گوس $2m$ (ξ_i, w_i) معمول فرض شده و لذا برای تابعی با درجه (order) $2m-1$ قابل قبول است.

$$m=1 \rightarrow 2m-1=1 \Rightarrow f: O(1)$$

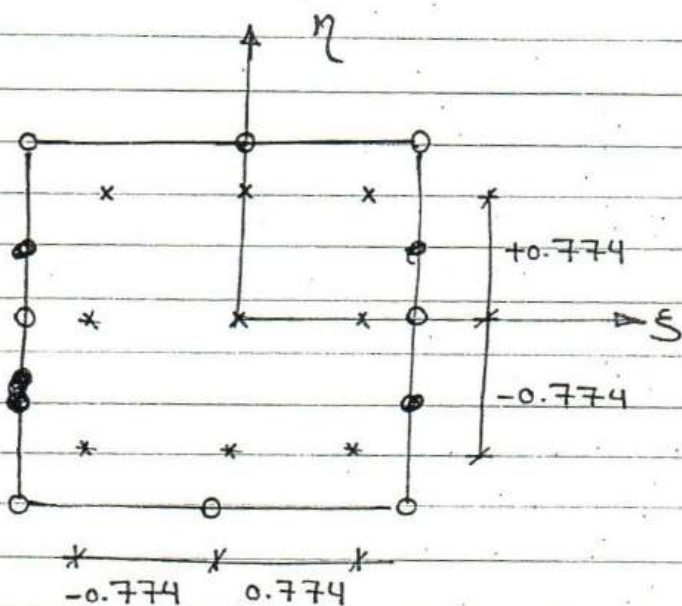


Subject

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Subject

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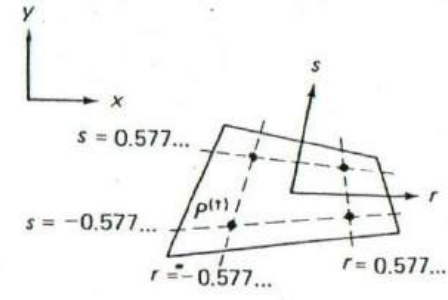
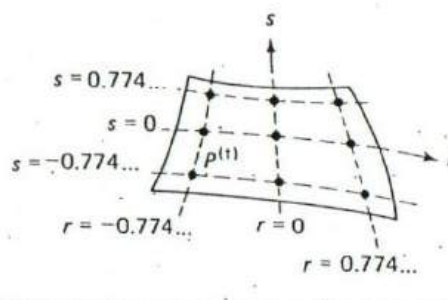
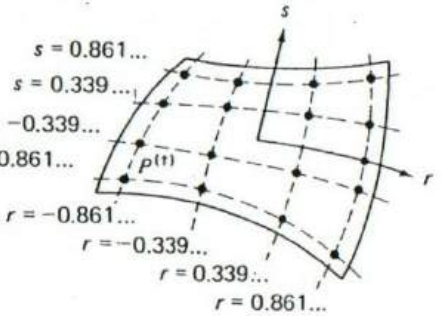


$N: 0(3) \rightarrow B: 0(2) \rightarrow P: 0(4)$

ABSCISSAE AND WEIGHT COEFFICIENTS OF THE GAUSSIAN QUADRATURE FORMULA

$$\int_{-1}^1 f(x) dx = \sum_{j=1}^n H_j f(a_j),$$

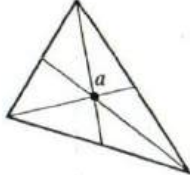
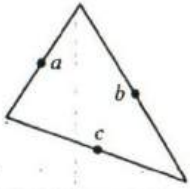
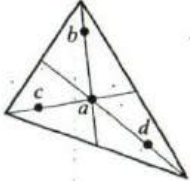
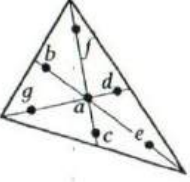
$\pm a$	n	H
0	$n = 1$	2.00000 00000 00000
0.57735 02691 89626	$n = 2$	1.00000 00000 00000
0.77459 66692 41483 0.00000 00000 00000	$n = 3$	0.55555 55555 55556 0.88888 88888 88889
0.86113 63115 94053 0.33998 10435 84856	$n = 4$	0.34785 48451 37454 0.65214 51548 62546
0.90617 98459 38664 0.53846 93101 05683 0.00000 00000 00000	$n = 5$	0.23692 68850 56189 0.47862 86704 99366 0.56888 88888 88889
0.93246 95142 03152 0.66120 93864 66265 0.23861 91860 83197	$n = 6$	0.17132 44923 79170 0.36076 15730 48139 0.46791 39345 72691
0.94910 79123 42759 0.74153 11855 99394 0.40584 51513 77397 0.00000 00000 00000	$n = 7$	0.12948 49661 68870 0.27970 53914 89277 0.38183 00505 05119 0.41795 91836 73469
0.96028 98564 97536 0.79666 64774 13627 0.52553 24099 16329 0.18343 46424 95650	$n = 8$	0.10122 85362 90376 0.22238 10344 53374 0.31370 66458 77887 0.36268 37833 78362
0.96816 02395 07626 0.83603 11073 26636 0.61337 14327 00590 0.32425 34234 03809 0.00000 00000 00000	$n = 9$	0.08127 43883 61574 0.18064 81606 94857 0.26061 06964 02935 0.31234 70770 40003 0.33023 93550 01260
0.97390 65285 17172 0.86506 33666 88985 0.67940 95682 99024 0.43339 53941 29247 0.14887 43389 81631	$n = 10$	0.06667 13443 08688 0.14945 13491 50581 0.21908 63625 15982 0.26926 67193 09996 0.29552 42247 14753

Integration order	Degree of precision	Location of integration points
2×2	3	
3×3	5	
4×4	7	

⁽¹⁾ The location of any integration point in the x - y coordinate system is given by: $x_p = \sum_i h_i(r_p, s_p)x_i$ and $y_p = \sum_i h_i(r_p, s_p)y_i$

TABLE 5.3 Gauss numerical integrations over rectangular domains. The integration weights are given in Table 5.2 using (5.132).

TABLE 8.2
NUMERICAL INTEGRATION FORMULAE FOR TRIANGLES

Order	Fig.	Error	Points	Triangular Co-ordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
Quadratic		$R = O(h^3)$	a b c	$\frac{1}{2}, \frac{1}{2}, 0$ $0, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{6}$
Cubic		$R = O(h^4)$	a b c d	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $0.6, 0.2, 0.2$ $0.2, 0.6, 0.2$ $0.2, 0.2, 0.6$	$\frac{27}{48}$ $\frac{2}{8}$ $\frac{2}{8}$ $\frac{2}{8}$
Quintic		$R = O(h^6)$	a b c d e f g	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $\alpha_1, \beta_1, \beta_1$ $\beta_1, \alpha_1, \beta_1$ $\beta_1, \beta_1, \alpha_1$ $\alpha_2, \beta_2, \beta_2$ $\beta_2, \alpha_2, \beta_2$ $\beta_2, \beta_2, \alpha_2$	0.22500, 0.0000 0.13239, 41527 0.12593, 91805

with
 $\alpha_1 = 0.0597158717$
 $\beta_1 = 0.4701420641$
 $\alpha_2 = 0.7974269853$
 $\beta_2 = 0.1012865073$

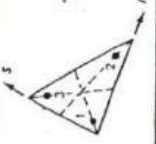
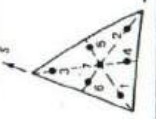
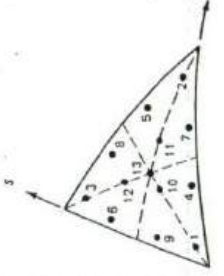
Integration order	Degree of precision	Integration points	r-coordinates	s-coordinates	Weights
3-point	2		$r_1 = 0.16666666666667$ $r_2 = 0.66666666666667$ $r_3 = r_1$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$	$w_1 = 0.33333333333333$ $w_2 = w_1$ $w_3 = w_1$
7-point	5		$r_1 = 0.1012865073235$ $r_2 = 0.7974269853531$ $r_3 = r_1$ $r_4 = 0.4701420641051$ $r_5 = r_4$ $r_6 = 0.0597158717898$ $r_7 = 0.3333333333333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_4$ $s_7 = r_7$	$w_1 = 0.1259391805448$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.1323941527885$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = 0.225$
13-point	7		$r_1 = 0.0651301029022$ $r_2 = 0.8697397941956$ $r_3 = r_1$ $r_4 = 0.3128654960049$ $r_5 = 0.6384441885698$ $r_6 = 0.0486903154253$ $r_7 = r_5$ $r_8 = r_4$ $r_9 = r_6$ $r_{10} = 0.2603459660790$ $r_{11} = 0.4793080678419$ $r_{12} = r_{10}$ $r_{13} = 0.3333333333333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_5$ $s_7 = r_6$ $s_8 = r_5$ $s_9 = r_4$ $s_{10} = r_{10}$ $s_{11} = r_{10}$ $s_{12} = r_{11}$ $s_{13} = r_{13}$	$w_1 = 0.0533472356088$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.0771137608903$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = w_4$ $w_8 = w_4$ $w_9 = w_4$ $w_{10} = 0.1756152574332$ $w_{11} = w_{10}$ $w_{12} = w_{10}$ $w_{13} = -0.149570044467$

TABLE 5.4 Gauss numerical integrations over triangular domains; here $\iint F(r,s)drds = \sum w_i F(r_i, s_i)$.

Similar extension for tetrahedra could obviously be made. Table 8.3 presents some such formulae based on reference 15.