

THEORY OF STABILITY

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سرفصل مطالب:

- ۱ - کمانش ارتجاعی و "ارتجاعی - خمیری" ستونها: باراولر، طول مؤثر، تئوری مدول دوگانه و مماسی، تئوری شنلی، ستونها با نقص اولیه، نحوه استفاده از این اصول در تدوین آیین نامه ها
- ۲ - روشهای تقریبی و کاربرد آنها در حل مسائل پایداری، بار بحرانی با استفاده از منحنی تغییر شکل تقریبی، انرژی پتانسیل ایستا، روش "رایلی - ریتز" و روش گلوکین
- ۳ - تیر ستونها، بررسی بارگذاریهای مختلف، تأثیر نیروی محوری بر روی سختی خمشی، مقاومت نهایی، نحوه استفاده از اصول در تدوین آیین نامه ها
- ۴ - کمانش پیچشی و پیچش جانبی اعضا، کمانش جانبی تیرهای با مقطع مستطیل در خمش خالص، کمانش جانبی تیرهای "Z" شکل، نحوه استفاده از این اصول در تدوین آیین نامه ها
- ۵ - کمانش تابها: بررسی بارگذاریهای مختلف، تأثیر نیروی محوری بر روی سختی خمشی، مقاومت قابها، نحوه استفاده از این اصول در تدوین آیین نامه ها.

مراجع:

1-Background to Buckling, Alen & Bulson

2-Stability of Structures: Principles and Applications Chai H. Yoo & Sung C. LEE

۳- پایداری سازه ها : الکساندر چاجس ترجمه: علی کاوه و علی برخورداری

4- Theory of Elastic Stability, Timoshenko & Gere

5- Beam-Column Theory Chen & Atsuta

ارزیابی:

تکالیف: ۱۰٪

امتحان اول ۴۰٪

امتحان دوم ۵۰٪

Introduction

- In discussing the analysis and design of various structures, we had two primary concerns:
 - the strength of the structure, i.e. its ability to support a specified load without experiencing excessive stresses;
 - the ability of the structure to support a specified load without undergoing unacceptable deformations.

Introduction

- Now we shall be concerned with stability of the structure,
 - with its ability to support a given load without experiencing a sudden change in its configuration.
- Our discussion will relate mainly to columns,
 - the analysis and design of vertical prismatic members supporting axial loads.
- Structures may fail in a variety of ways, depending on the :
 - Type of structure
 - Conditions of support
 - Kinds of loads
 - Material used

Introduction

- Failure is prevented by designing structures so that the maximum stresses and maximum displacements remain within tolerable limits.
- Strength and stiffness are important factors in design as we have already discussed
- Another type of failure is buckling
- If an element length is an order of magnitude large than either of its other dimensions and is under a compressive load, it is called a **columns**.
- Due to its size its axial displacement is going to be very small compared to its lateral deflection called *buckling*.



Introduction

- Quite often the buckling of column can lead to sudden and dramatic failure. And as a result, special attention must be given to design of column so that they can safely support the loads.
- Buckling is not limited to columns.
 - Can occur in many kinds of structures
 - Can take many forms
 - Step on empty aluminum can
 - Major cause of failure in structures



Critical Load

- This is analogous to a ball placed on a smooth surface
 - If the surface is concave (inside of a dish) the equilibrium is stable and the ball always returns to the low point when disturbed
 - If the surface is convex (like a dome) the ball can theoretically be in equilibrium on the top surface, but the equilibrium is unstable and the ball rolls away
 - If the surface is perfectly flat, the ball is in neutral equilibrium and stays where placed.

Critical Load

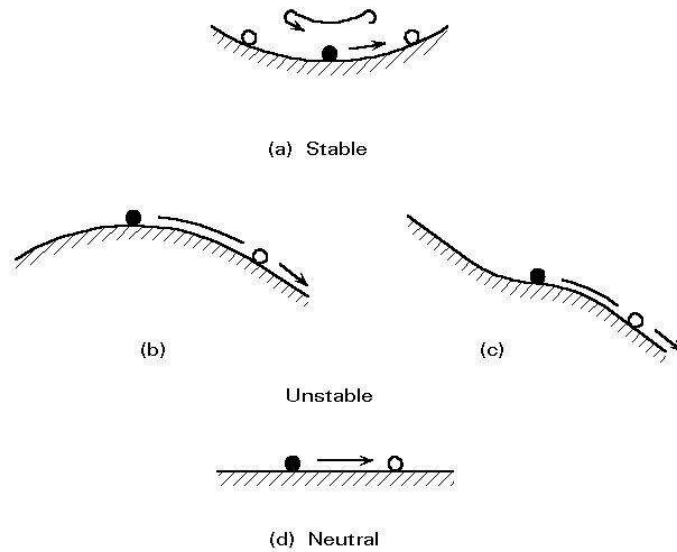
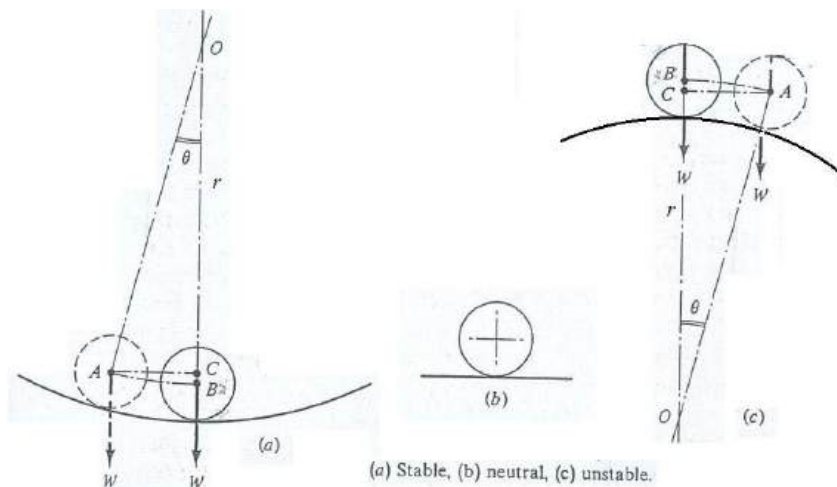


Figure 1 The three states of equilibrium



$$V = V_0 + wr(1 - \cos \theta)$$

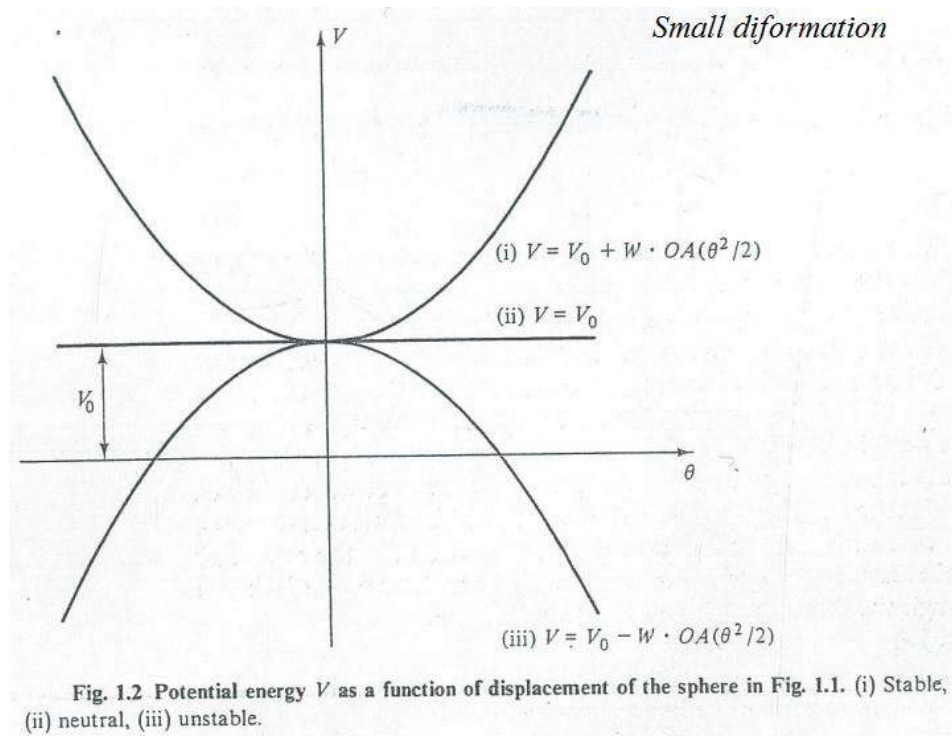
$$\text{equilibrium: } \frac{dV}{d\theta} = +wr \sin \theta = 0 \Rightarrow \theta = 0$$

$$\text{stable equilibrium: } \left. \frac{d^2V}{d\theta^2} \right|_{\theta=0} = +wr > 0$$

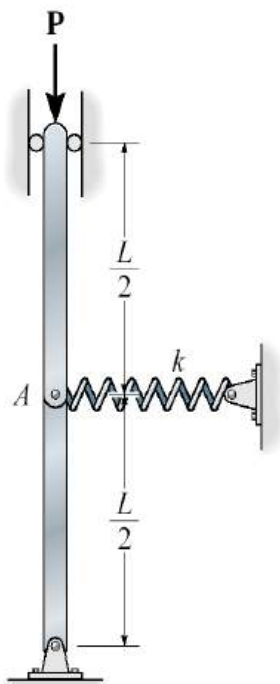
$$V = V_0 - wr(1 - \cos \theta)$$

$$\text{equilibrium: } \frac{dV}{d\theta} = -wr \sin \theta = 0 \Rightarrow \theta = 0$$

$$\text{Unstable equilibrium: } \left. \frac{d^2V}{d\theta^2} \right|_{\theta=0} = -wr < 0$$



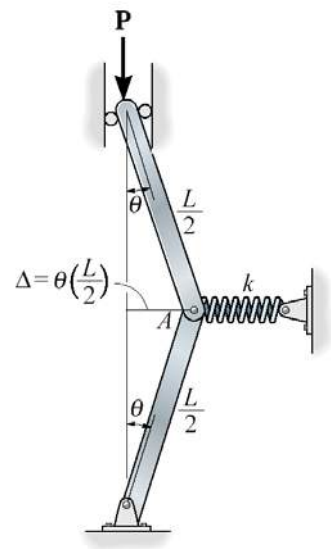
Buckling & Stability



- Consider the figure, Hypothetical structure
- Two rigid bars joined by a pin the center, held in a vertical position by a spring
- Both end have simple supports at the end and are compressed by an axial load P .
- Elasticity of the buckling model is concentrated in the spring (real model can bend throughout its length)
- Two bars are perfectly aligned
- Load P is along the vertical axis
- Spring is unstressed
- Bar is in direct compression

Buckling & Stability

- Structure is disturbed by an external force that causes point A to move a small distance laterally.
- Rigid bars rotate through small angles θ
- Force develops in the spring
- Direction of the force tends to return the structure to its original straight position, called the Restoring Force.
- At the same time, the tendency of the axial compressive force is to increase the lateral displacement.
- These two actions have opposite effects
 - Restoring force tends to decrease displacement
 - Axial force tends to increase displacement.



Critical Load

- Now remove the disturbing force:
- If **P is small**, the restoring force will dominate over the action of the axial force and the structure will return to its initial straight position, in this case Structure is called **Stable**.
- **if P is large**, the lateral displacement of A will increase and the bars will rotate through larger and larger angles until the structure collapses, in this case Structure is **unstable** and fails by lateral buckling
- Transition between stable and unstable conditions occurs at a value of the axial force called the **Critical Load** P_{cr} .

Critical Load

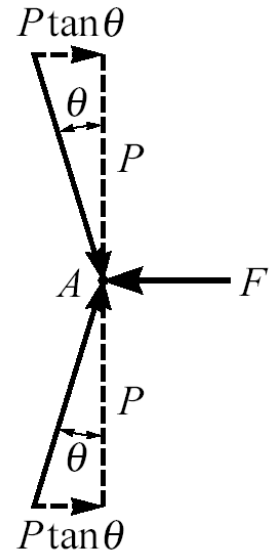
Find the critical load by considering the structure in the disturbed position and consider equilibrium

- Consider the entire structure as a FBD and sum the forces in the x direction
- Next, consider the upper bar as a free body. Subjected to axial forces P and force F in the spring. Force is equal to the stiffness k times the displacement Δ , $F = k\Delta$. Since θ is small, the lateral displacement of point A is $\theta L/2$. Applying equilibrium and solving for P : $P_{cr} = kL/4$

$$2P \tan \theta = 2P\theta = 4P\Delta / L = F = K\Delta$$

$$\text{trivial: } \Delta = 0$$

$$\text{nontrivial: } P_{cr} = KL/4$$



Critical Load

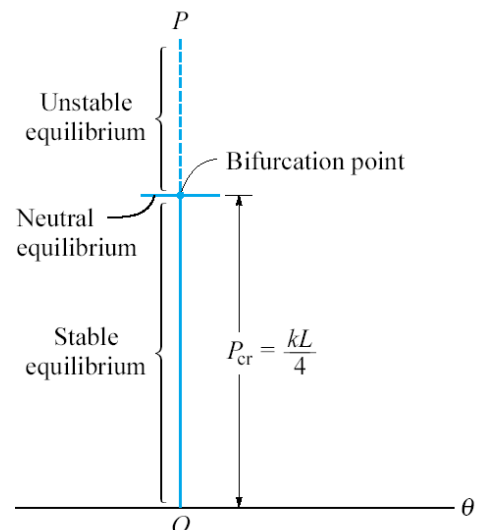
- Which is the critical load
 - At this value the structure is in equilibrium regardless of the magnitude of the angle (provided it stays small)
 - Critical load is the only load for which the structure will be in equilibrium in the disturbed position
 - At this value, restoring effect of the moment in the spring matches the buckling effect of the axial load
 - Represents the boundary between the stable and unstable conditions.

Critical Load

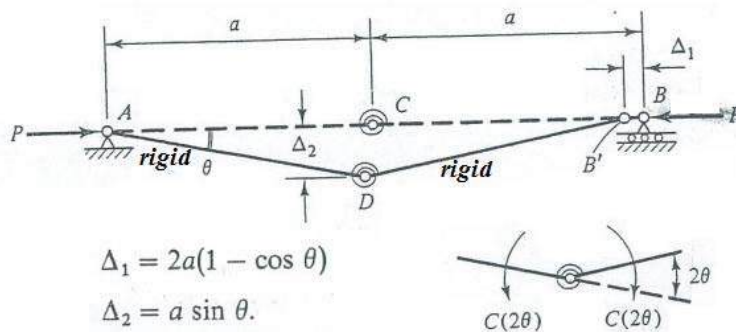
- If the axial load is less than P_{cr} the effect of the moment in the spring dominates and the structure returns to the vertical position after a small disturbance – stable condition.
- If the axial load is larger than P_{cr} the effect of the axial force predominates and the structure buckles – unstable condition.

Critical Load

- The boundary between stability and instability is called neutral equilibrium.
- The critical point, after which the deflections of the member become very large, is called the "bifurcation point" of the system.



مطلوب است بحث در رابطه با پایداری سیستم ۱ درجه آزادی شکل زیر:
 دو میله صلب هر یک در یک انتها به تکیه گاه مفصلی و در انتهای دیگر به یکدیگر مفصل و دوران
 آنها با یک فنر پیچشی به سختی C در گیر شده است.



الف- تغییر شکل های بزرگ:

$$\Delta_1 = 2a(1 - \cos \theta)$$

$$\Delta_2 = a \sin \theta.$$

$$U = \frac{1}{2}C(2\theta)(2\theta) = 2C\theta^2.$$

$$V_p = -P\Delta_1.$$

$$V = U + V_p = 2C\theta^2 - 2Pa(1 - \cos \theta).$$

$$\frac{dV}{d\theta} = 4C\theta - 2Pa \sin \theta = 0 \Rightarrow P = \frac{2C}{a} \cdot \frac{\theta}{\sin \theta}$$

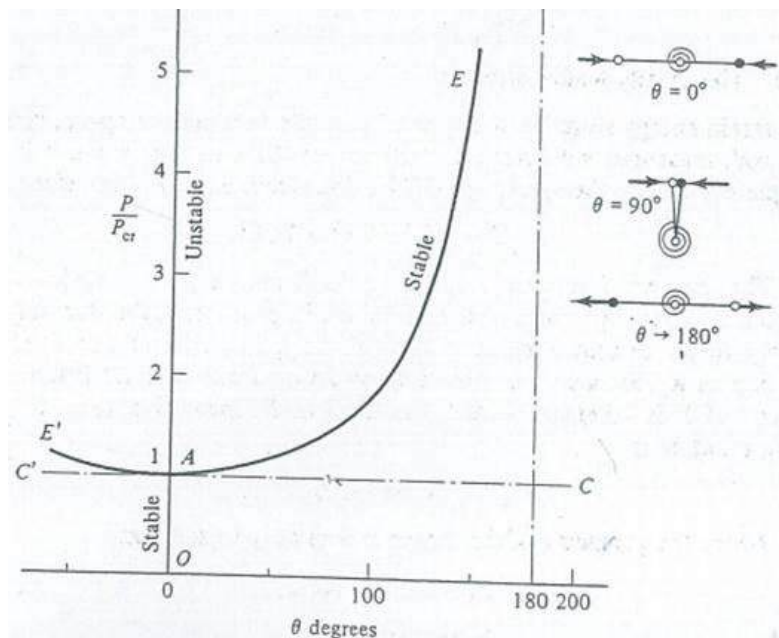
$$\frac{d^2V}{d\theta^2} = 4C - 2Pa \cos \theta. \quad \left| \begin{array}{l} = 4C(1 - \frac{\theta}{\tan \theta}) > 0 \\ P = \frac{2C}{a} \cdot \frac{\theta}{\sin \theta} \end{array} \right.$$

انرژی پتانسیل ذخیره شده در فنر:
 از دست دادن انرژی پتانسیل:

انرژی پتانسیل کل:

صفر شدن مشتق انرژی پتانسیل کل (تعادل):

مثبت بودن مشتق دوم انرژی پتانسیل (تعادل پلیدار):



الف- تغییر شکل های کوچک:

$$\sin \theta = \theta; \quad \cos \theta = 1 - \frac{\theta^2}{2}$$

انرژی پتانسیل کل:

$$V = U + V_p = \frac{1}{2}C(2\theta)^2 - P\Delta_1$$

انرژی پتانسیل کل:

$$V = 2C\theta^2 - Pa\theta^2$$

$$\frac{dV}{d\theta} = (4C - 2Pa)\theta = 0 \quad P_{cr} = \frac{2C}{a}$$

$$\frac{d^2V}{d\theta^2} = 4C - 2Pa \begin{array}{ll} > 0 & \text{if } P < P_{cr} \\ < 0 & \text{if } P > P_{cr} \end{array}$$

صفر شدن مشتق انرژی پتانسیل کل (تعادل):

مثبت بودن مشتق دوم انرژی پتانسیل (تعادل پلیدار):
 منفی بودن مشتق دوم انرژی پتانسیل (تعادل ناپلیدار):

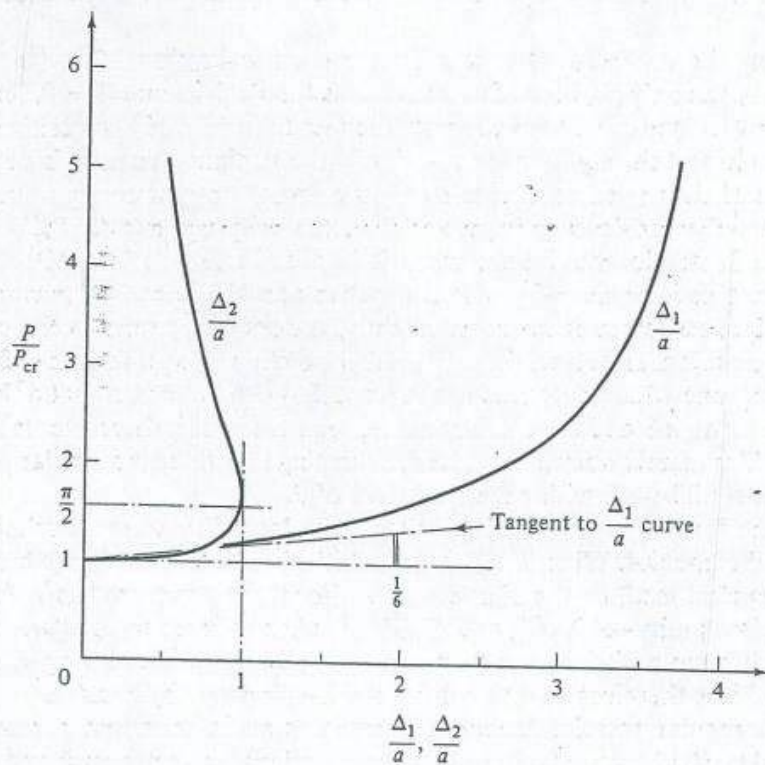


Fig. 1.7 Relationship between axial force and (a) longitudinal displacement Δ_1 and (b) transverse displacement Δ_2 for the structure in Fig. 1.5.

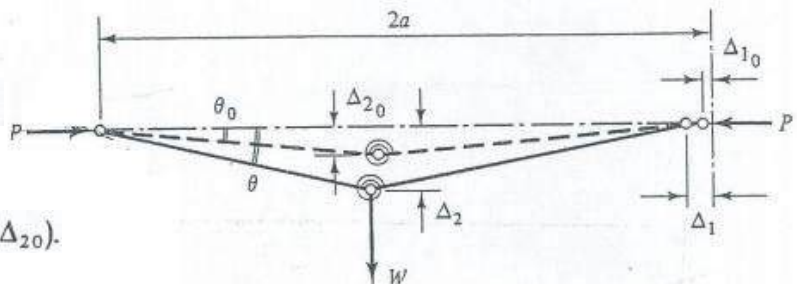
مثال قبل را اگر نقص اولیه Δ_{02} و نیروی جانبی W را در وسط تیر داشته باشد بررسی کنید.

انرژی پتانسیل ذخیره شده در فنر:

$$U = \frac{C}{2} \{2(\theta - \theta_0)\}^2 = 2C(\theta - \theta_0)^2.$$

از دست دادن انرژی پتانسیل:

$$V_p = -P(\Delta_1 - \Delta_{10}) - W(\Delta_2 - \Delta_{20}).$$



$$V = U + V_p = 2C(\theta - \theta_0)^2 + 2Pa (\cos \theta - \cos \theta_0) - Wa (\sin \theta - \sin \theta_0).$$

انرژی پتانسیل کل:

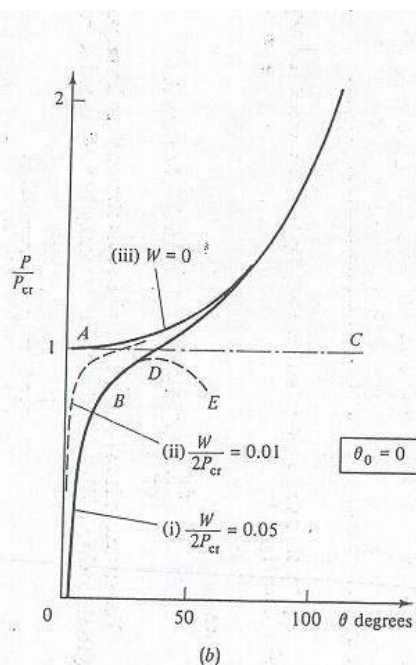
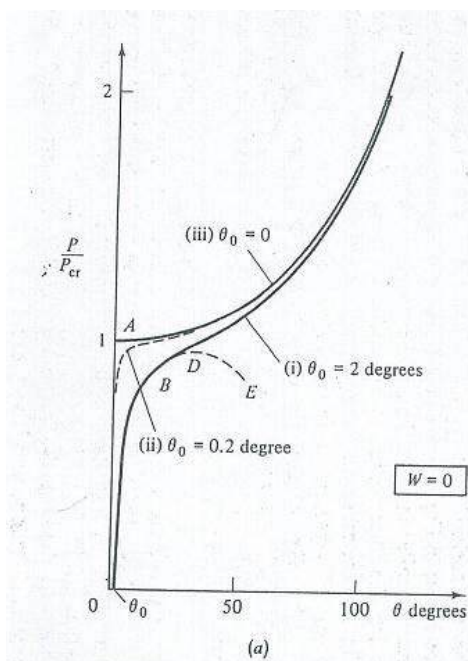
صفر شدن مشتق انرژی پتانسیل کل (تعادل):

$$\frac{dV}{d\theta} = 4C(\theta - \theta_0) - 2Pa \sin \theta - Wa \cos \theta.$$

$$\frac{dV}{d\theta} = 0. \quad \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin \theta} - \frac{W}{2P_{cr}} \cot \theta.$$

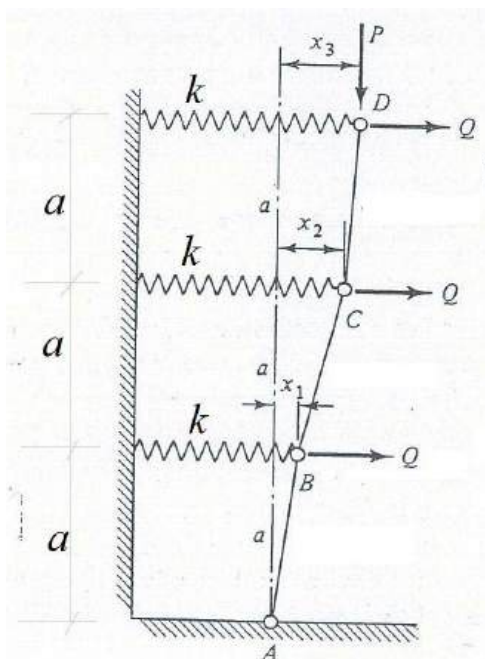
$$\frac{d^2V}{d\theta^2} = 4C - 2Pa \cos \theta + Wa \sin \theta = 4C \left(1 - \frac{P}{P_{cr}} \cos \theta + \frac{W}{2P_{cr}} \sin \theta \right)$$

$$= 4C(1 - (\theta - \theta_0) \cot \theta) + \frac{W}{P_{cr}} \left(\frac{1}{2} \frac{1 - 2 \cos^2 \theta}{\sin \theta} \right)$$



نمودار نیرو-تغییر مکان سیستم a (اثرات نقص اولیه θ_a) b (اثرات بارهای جانبی

مثال- سیستم سه درجه آزادی شکل زیر متشکل از سه میله صلب که به هم مفصل شده اند و توسط یک تکیه گاه ساده و سه فنر به سختی k مقید گردیده است در نظر بگیرید. با فرض کوچک بودن تغییر شکل ها در رابطه با پایداری سیستم بحث کنید.

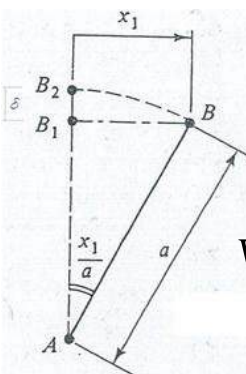


$$U = \frac{k}{2} (x_1^2 + x_2^2 + x_3^2).$$

انرژی پتانسیل ذخیره شده در فنر ها:

$$V_p = -P \times (\text{descent of } D)$$

از دست دادن انرژی پتانسیل توسط نیروی P :



$$Q = 0$$

$$\delta = B_2 B_1 = a(1 - \cos(x_1/a)) \approx a(1 - 1 + (x_1/a)^2 / 2) = x_1^2 / 2a$$

$$V_p = -P \left[\frac{x_1^2}{2a} + \frac{(x_2 - x_1)^2}{2a} + \frac{(x_3 - x_2)^2}{2a} \right]$$

انرژی پتانسیل کل:

$$V = U + V_p + V_q = \frac{k}{2} (x_1^2 + x_2^2 + x_3^2)$$

$$- \frac{P}{2a} (2x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3) - Q(x_1 + x_2 + x_3)$$

صفر شدن مشتقات جزئی انرژی پتانسیل کل نسبت به درجات آزادی مجهول (تعال):

$$\frac{\partial V}{\partial x_1} = kx_1 - \frac{P}{2a} (4x_1 - 2x_2) \quad -Q = 0;$$

$$\frac{\partial V}{\partial x_2} = kx_2 - \frac{P}{2a} (4x_2 - 2x_1 - 2x_3) \quad -Q = 0;$$

$$\frac{\partial V}{\partial x_3} = kx_3 - \frac{P}{2a} (2x_3 - 2x_2) \quad -Q = 0.$$

$$\begin{bmatrix} (1-2p) & p & 0 \\ p & (1-2p) & p \\ 0 & p & (1-p) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{Q}{k} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad p = P/ka$$

در صورت صرف نظر کردن از بارهای جانبی دسته معادلات تعادل به صورت زیر ساده میشوند:

$$\begin{bmatrix} (1-2p) & p & 0 \\ p & (1-2p) & p \\ 0 & p & (1-p) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$x_1 = x_2 = x_3 = 0$$

جواب بدیهی معادلات فوق:

جواب غیر بدیهی معادلات فوق از صفر کردن دترمینان ضرایب حاصل میشود که به معنی تعیین نیروی P به قسمی است که به ازای آن جواب دسته معادلات فوق دیگر یکتا نباشد. که همان بار بحرانی است:

$$p^3 - 6p^2 + 5p - 1 = 0.$$

$$p_1 = 0.3080, \quad p_2 = 0.6431, \quad p_3 = 5.049.$$

از قرار دادن p مربوط به هر بار بحرانی در رابطه زیر مود کمانشی مربوط به آن بار بحرانی حاصل میگردد.

$$\begin{bmatrix} (1-2p) & p & 0 \\ p & (1-2p) & p \\ 0 & p & (1-p) \end{bmatrix} \begin{Bmatrix} 1 \\ x_2/x_1 \\ x_3/x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{x_2}{x_1} = -1.25; \quad \frac{x_3}{x_1} = +0.555.$$

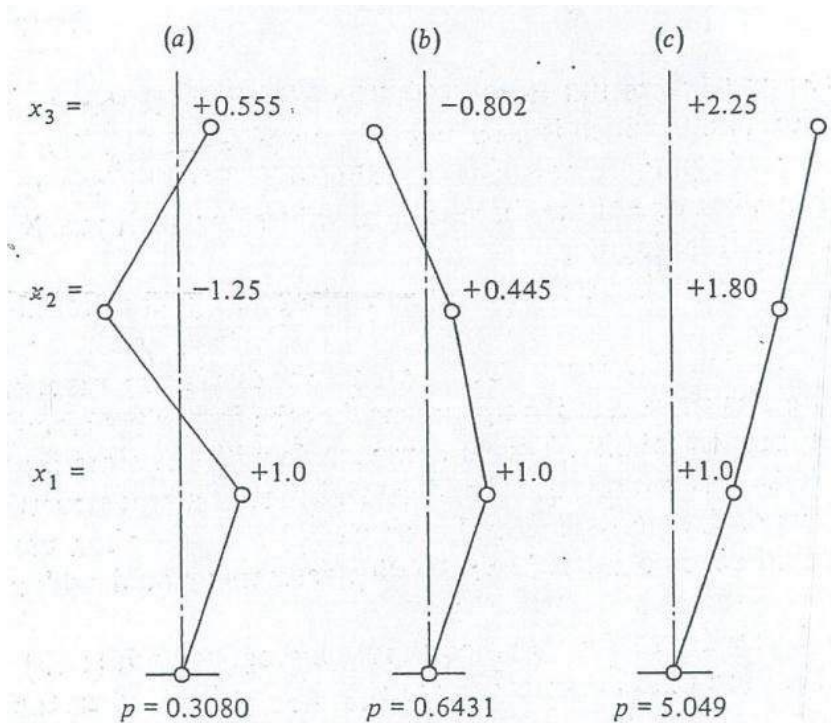


Fig. 1.20 Buckling modes for the system in Fig. 1.19.

سازه های پیوسته:

معادله دیفرانسیل حاکم با استفاده از اصل انرژی پتانسیل ایستا:

انرژی کرنشی ذخیره ناشی از
تغییر شکل های محوری و خمشی:

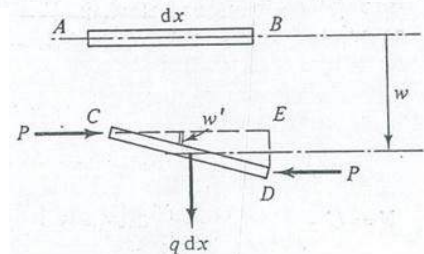
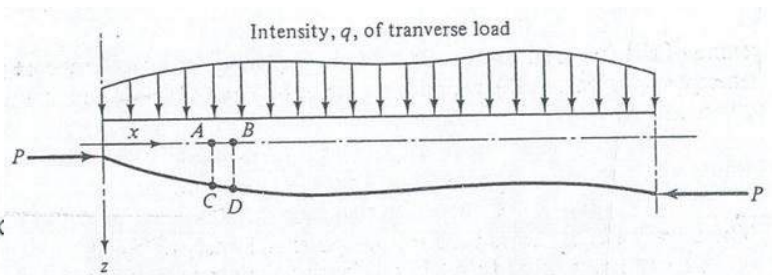
$$U = \int_0^L \frac{EI}{2} (w'')^2 dx + \int_0^L \frac{P^2}{2EA} dx$$

از دست دادن انرژی پتانسیل:

$$V_p = -\frac{P}{2} \int_0^L w'^2 dx - \int_0^L \frac{P^2}{EA} dx. \quad \text{نیروی محوری:}$$

$$V_q = -\int_0^L wq dx.$$

بارهای جانبی:



انرژی پتانسیل کل:

$$V = U + V_p + V_q = \int_0^L \frac{EI}{2} (w'')^2 dx - \frac{P}{2} \int_0^L (w')^2 dx - \int_0^L wq dx - \int_0^L \frac{P^2}{2EA} dx.$$

ایستا کردن انرژی پتانسیل کل با استفاده از رابطه اولر:

$$V = \int_{x_1}^{x_2} F(x, w, w', w'') dx$$

ایستا کردن تابع روبرو:

$$F(x, w, w', w'') = \frac{EI}{2} (w'')^2 - \frac{P}{2} (w')^2 - wq - \frac{P^2}{2EA}$$

$$\frac{d^2}{dx^2} \left(\frac{\partial F}{\partial w''} \right) - \frac{d}{dx} \left(\frac{\partial F}{\partial w'} \right) + \frac{\partial F}{\partial w} = 0$$

به شرط صدق کردن در رابطه اولر:

$$\frac{d^2}{dx^2} (EIw'') + \frac{d}{dx} (Pw') - q = 0$$

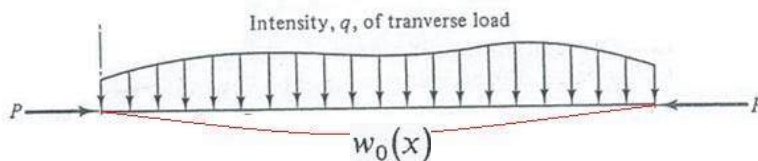
معادله دیفرانسیل حاکم بر تعادل:

$$EIw^{IV} + Pw'' = q$$

در صورت ثابت بودن EI و P :

$$w = c_1 \sin \mu x + c_2 \cos \mu x + c_3 x + c_4 + f(x)$$

وجود نقص اولیه $w_0(x)$ در ستون قبل از بارگذاری:



انرژی پتانسیل کل:

$$V = \int_0^L \left[\frac{EI}{2} (w'' - w_0'')^2 - \frac{P}{2} (w' - w_0')^2 - (w - w_0)q - \frac{P^2}{2EA} \right] dx$$

معادله دیفرانسیل حاکم بر تعادل حاصل از انرژی پتانسیل کل: $(EIw'')'' + Pw'' = q + (EIw_0'')''$

$$w^{IV} + \mu^2 w'' = \frac{q}{EI} + w_0^{IV}$$

در صورت ثابت بودن EI و P :

$$w = c_1 \sin \mu x + c_2 \cos \mu x + c_3 x + c_4 + f(x)$$

جواب معادله دیفرانسیل حاکم:

مثال - ستون با تکیه گاههای ساده $EIw^{iv} + Pw'' = 0$

$$w = C_1 \sin \mu x + C_2 \cos \mu x + C_3 x + C_4$$

$$w'' = -c_1 \mu^2 \sin \mu x - c_2 \mu^2 \cos \mu x.$$

تعیین ثابت ها با استفاده از شرایط مرزی:
در ابتدا و انتهای ستون ممان و تغییر مکان صفر است

$$@ x=0 \quad w=0 \quad C_2 + C_4 = 0$$

$$@ x=0 \quad w''=0 \quad C_2 = 0 \Rightarrow C_4 = 0$$

$$@ x=l \quad w=0 \quad C_1 \sin \mu l + C_3 l = 0$$

$$@ x=l \quad w''=0 \quad -C_1 \mu^2 \sin \mu l = 0 \Rightarrow C_3 = 0$$

$$C_1 \sin \mu l = 0$$

$$C_1 = 0$$

جواب غیر بدیهی که از صفر کردن دترمینان ضرایب حاصل میشود

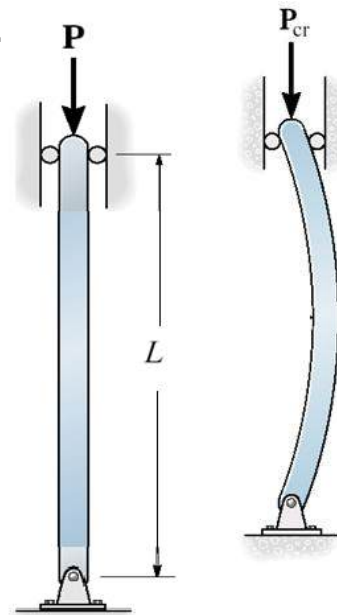
$$\sin \mu l = 0 \Rightarrow \mu l = i\pi$$

$$w = C_1 \sin \mu x = C_1 \sin \frac{i\pi x}{L}$$

مودهای کمانشی:

$$P_{cri} = \frac{i^2 \pi^2 EI}{L^2}$$

بارهای کمانشی:



جواب بدیهی :

شکل دادن دترمینان ضرایب:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin \mu L & \cos \mu L & L & 1 \\ 0 & 1 & 0 & 0 \\ \sin \mu L & \cos \mu L & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$L \sin \mu L = 0 \Rightarrow \mu L = i\pi$$

$$P = i^2 \pi^2 EI / L^2 \quad (i = 1, 2, 3, \dots)$$

$$w = c_1 \sin \mu x \quad \text{or} \quad w = c_1 \sin \frac{i\pi x}{L}$$

First mode of buckling $P_1 = \frac{\pi^2 EI}{L^2}$

Second mode of buckling $P_2 = \frac{4\pi^2 EI}{L^2}$

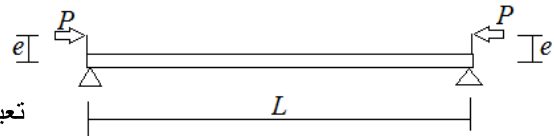
Third mode of buckling $P_3 = \frac{9\pi^2 EI}{L^2}$

شکل مودهای کمانشی:

مثال - ستون با تکیه گاههای ساده تحت اثر بارهای محوری خارج از مرکز در دو انتها

$$EIw^{iv} + Pw'' = 0$$

$$w = C_1 \cos \mu x + C_2 \sin \mu x + C_3 x + C_4$$



تعیین ثابت ها با استفاده از شرایط مرزی:
در ابتدا و انتهای ستون مماس Pe و تغییر مکان صفر است

$$@ x=0 \quad w=0 \quad C_1 + C_4 = 0$$

$$@ x=0 \quad -EIw'' = Pe \Rightarrow C_1 \mu^2 = \mu^2 e \Rightarrow C_1 = e, \quad C_4 = -e$$

$$@ x=L \quad w=0 \quad e \cos \mu L + C_2 \sin \mu L + C_3 L - e = 0$$

$$@ x=L \quad -EIw'' = Pe \Rightarrow e \mu^2 \cos \mu L - C_2 \mu^2 \sin \mu L = \mu^2 e \Rightarrow C_2 = -e(\cos \mu L - 1) / \sin \mu L$$

$$0 = C_3$$

$$w = e(\cos \mu x + (1 - \cos \mu L) \sin \mu x / \sin \mu L - 1)$$

جواب بدیهی :

جواب غیر بدیهی که از صفر کردن دترمینان ضرایب حاصل میشود $\cos \mu L / 2 = 0 \Rightarrow \mu L / 2 = (2i+1)\pi / 2$

$$P_{cri} = \frac{i^2 \pi^2 EI}{L^2} \quad i = 1, 3, 5, 7, 9, \dots$$

بارهای کمانشی:

مثال - ستون با تکیه گاه غلتکی و یک تکیه گاه گیردار تحت اثر بار محوری :

$$EIw^{iv} + Pw'' = 0$$

$$w = C_1 \cos \mu x + C_2 \sin \mu x + C_3 x + C_4$$



شرایط مرزی: در ابتدا ستون مماس و تغییر مکان صفر است
و انتهای شیب و تغییر مکان صفر است.

$$w = w' = 0 \quad \text{at} \quad x = 0;$$

$$w = w'' = 0 \quad \text{at} \quad x = L.$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \mu & 0 & 1 & 0 \\ \sin \mu L & \cos \mu L & L & 1 \\ \sin \mu L & \cos \mu L & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

با صفر کردن دترمینان ضرایب ماتریس فوق معادله مشخصه تعیین بار بحرانی حاصل میگردد:

$$\mu L = \tan \mu L,$$

کوچکترین جواب معادله بالا $\mu L = 4.49$ باشد و بار بحرانی مربوطه:

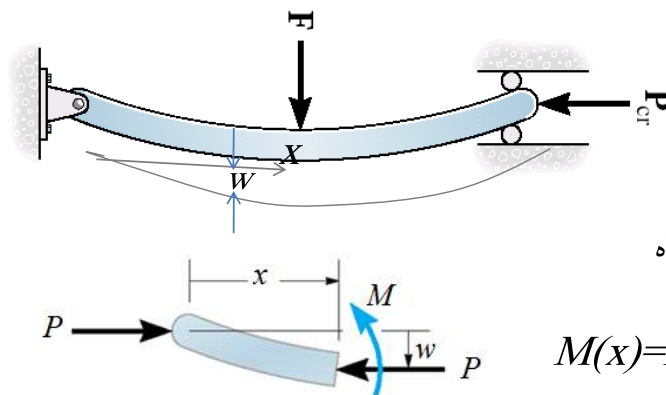
$$P_1 = (\mu^2 EI)_{crit} = \frac{(4.49)^2}{L^2} EI = \frac{2.05 \pi^2 EI}{L^2}$$

و مود کمانشی مربوطه به صورت زیر حاصل میگردد:

$$w = c_1 \{ \sin \mu x - \mu x + \mu L (1 - \cos \mu x) \}$$

تعیین اثرات نیروی محوری بر روی تغییر شکل ها و تعیین بار بحرانی با استفاده از روش تعادل:

- ابتدا تعیین با بحرانی ستونها در شرایط تکیه گاهی زیر به روش تعادل بررسی میشود:
- تیر با تکیه گاه های ساده
- تیر دو سر گیردار
- تیر کنسول



تیر با تکیه گاه های ساده

تعادل ممان در نقطه ای به فاصله x از بر تکیه گاه

$$M(x) = Pw$$

$$-EI \frac{d^2 w}{dx^2} = Pw \quad \text{Or:} \quad \frac{d^2 w}{dx^2} + \left(\frac{P}{EI} \right) w = 0$$

$$\mu^2 = \frac{P}{EI}$$

با توجه به ثابت و مثبت بودن نسبت P/EI :

$$\frac{d^2 w}{dx^2} + \mu^2 w = 0$$

معادله دیفرانسیل فوق به صورت مقابل ساده میشود:

جواب معادله دیفرانسیل فوق به صورت زیر خواهد بود:

$$w = A \cos(\mu x) + B \sin(\mu x)$$

بطوریکه مقادیر A و B با توجه به شرایط مرزی قابل محاسبه میباشند:

$$@ x=0, w=0, \text{ therefore } A=0$$

شرایط مرزی عبارتند از:

$$@ x=L, w=0, \text{ then } 0=B \sin(\mu L)$$

$$B=0,$$

جواب بدیهی:

$$\sin(\mu L)=0$$

جواب غیر بدیهی:

$$\mu L = n\pi \quad n=1,2,3,\dots$$

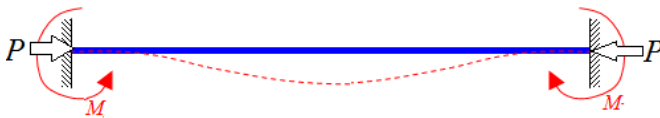
یعنی:

$$\mu^2 = \frac{P}{EI} = \left(\frac{n\pi}{L} \right)^2$$

• از آنجا که:

$$P = n^2 \frac{\pi^2 EI}{L^2}$$

• بار بحرانی به صورت زیر محاسبه میگردد:



ستون دو سر گیردار:

$$EIw'' + Pw = M$$

$$w = A \cos \mu x + B \sin \mu x + M/P$$

$$@ x=0 \quad w=0 \quad A + M/P = 0$$

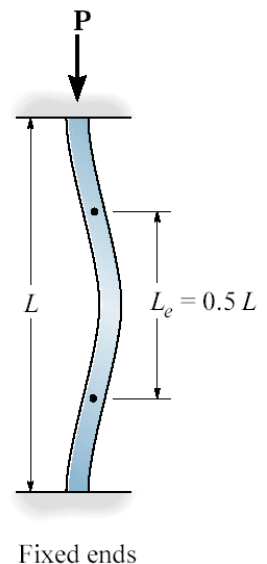
$$@ x=0 \quad w' = 0 \quad B = 0$$

$$@ x=L/2 \quad w' = 0 \quad A\mu \sin \mu L/2 = 0$$

$$\text{trivial sol. } A=0 \quad \text{Nontrivial sol. } \sin \mu L/2 = 0$$

$$\Rightarrow \mu L/2 = i\pi \Rightarrow P_{cri} = 4i^2 \pi^2 EI / L^2$$

- The critical load for other column can be expressed in terms of the critical buckling load for a pin-ended column.
- From symmetry conditions at the point of inflection occurs at $1/4 L$.
- Therefore the middle half of the column can be taken out and treated as a pin-ended column of length $L_e = L/2$
- Yielding:



ستون کنسول:

$$EIw'' + Pw = P\delta$$

$$w = A\cos \mu x + B\sin \mu x + \delta$$

$$@ x=0 \quad w=0 \quad A+\delta=0$$

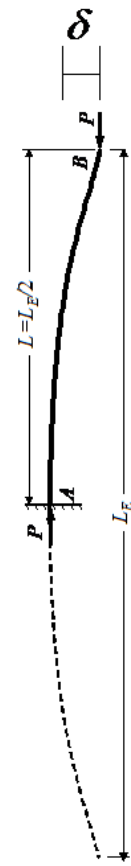
$$@ x=0 \quad w'=0 \quad B=0$$

$$@ x=L \quad w=\delta \quad A\cos \mu l = 0$$

$$\text{trivial sol. } A=0 \quad \text{Nontrivial sol. } \cos \mu l = 0$$

$$\Rightarrow \mu l = \pi / 2 \Rightarrow P_{cr1} = \pi^2 EI / 4l^2$$

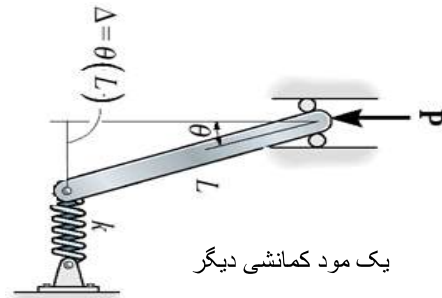
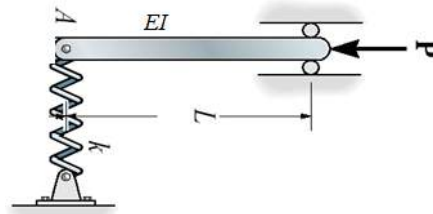
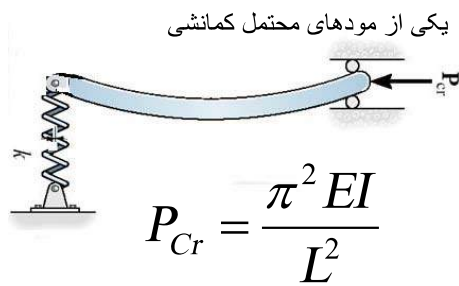
- The span effective length (L_e) is equivalent to $\frac{1}{2}$ of the Euler span



مفهوم طول موثر

$$L_e = \begin{cases} L & \text{pin - pin} \\ 0.7L & \text{fixed - pin} \\ 0.5L & \text{fixed - fixed} \\ 2L & \text{fixed - free} \end{cases}$$

تعیین بار کمانشی سازه زیر مورد نظر است



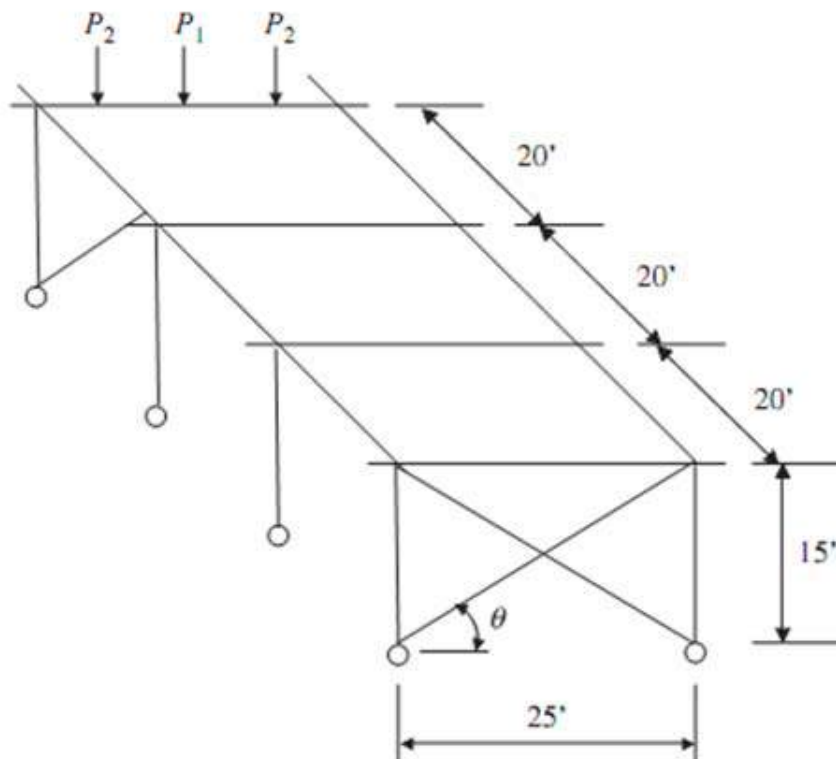
$$\sum M_o = 0 \Rightarrow P\Delta - kL\Delta = 0$$

$$(P - kL)\Delta = 0$$

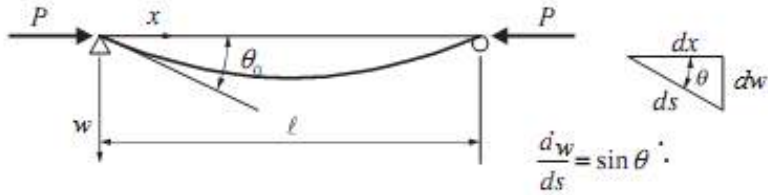
$$\therefore \text{trivial sol. } \Delta = 0$$

$$\therefore \text{nontrivial sol. } P_{cr} = kL$$

$$\text{Final Decition: } P_{Cr} = \begin{cases} \frac{\pi^2 EI}{L^2} & \text{for : } k > \frac{\pi^2 EI}{L^3} \\ kL & \text{for : } k < \frac{\pi^2 EI}{L^3} \end{cases}$$



تغییر شکل های بزرگ (تیر با تکیه گاههای ساده):



رابطه بین شعاع انحنا (ρ) و گشتاور خمشی:

$$M = -\frac{EI}{\rho}$$

رابطه شعاع انحنا:

$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

تعداد در مقطع تیر:

$$Pw = M = -\frac{EI}{\rho} = -EI \frac{d\theta}{ds}$$

معادله دیفرانسیل حاکم:

$$EI \frac{d\theta}{ds} + Pw = 0$$

و یا:

$$\frac{d\theta}{ds} + \mu^2 w = 0$$

با مشتق گیری نسبت به s از رابطه اخیر:

$$\frac{d^2\theta}{ds^2} + \mu^2 \frac{dw}{ds} = 0$$

از آنجا که در حالت تغییر شکل های بزرگ (با توجه به شکل):

$$\sin \theta = \frac{dw}{ds}$$

و معادله دیفرانسیل تعادل بر حسب متغیر های s و θ شکل میگیرد:

$$\frac{d^2\theta}{ds^2} + \mu^2 \sin \theta = 0$$

با ضرب معادله اخیر در $2d\theta$ و انتگرال گیری از آن:

$$\int 2 \frac{d^2\theta}{ds^2} \frac{d\theta}{ds} ds + \mu^2 \int 2 \sin \theta d\theta = 0$$

با در نظر گرفتن معادلات ریاضی زیر:

$$\frac{d}{ds} \left(\frac{d\theta}{ds} \right)^2 = 2 \frac{d^2\theta}{ds^2} \frac{d\theta}{ds} \quad d(\cos \theta) = -\sin \theta d\theta$$

معادله انتگرالی اخیر بصورت زیر نوشته میشود:

$$\int d\left(\frac{d\theta}{ds}\right)^2 - 2\mu^2 \int d(\cos \theta) = 0$$

با انتگرال گیری از رابطه بالا رابطه زیر حاصل میگردد:

$$\left(\frac{d\theta}{ds}\right)^2 - 2\mu^2 \cos \theta = C$$

با توجه انحنای صفر در ابتدای تیر:

$$\frac{d\theta}{ds} = 0 \quad \text{at } x = 0,$$

$$\left(\text{moment} = 0 \Rightarrow \frac{1}{\rho} = 0 \text{ or } \rho = \infty, \text{ straight line} \right) \text{ and } \theta = \theta_0$$

بنابراین ثابت انتگرالی بدست می آید:

$$C = -2\mu^2 \cos \theta_0$$

با جایگذاری C از رابطه بالا در معادله صفحه قبل:

$$\left(\frac{d\theta}{ds}\right)^2 - 2\mu^2(\cos \theta - \cos \theta_0) = 0$$

اگر از رابطه بالا جذر گرفته شود:

$$ds = -\frac{d\theta}{\sqrt{2\mu} \sqrt{\cos \theta - \cos \theta_0}}$$

$$P = \frac{4K^2}{\ell^2 EI} = \frac{4EI K}{\ell^2} \quad \text{and} \quad P_{cr} = \frac{\pi^2 EI}{\ell^2}$$

با انتگرال گیری از رابطه بالا رابطه زیر حاصل میگردد:

$$\int_0^{\ell/2} ds = -\frac{1}{\sqrt{2\mu}} \int_{\theta_0}^0 \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \text{ or } \frac{\ell}{2} = \frac{1}{\sqrt{2\mu}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}$$

or

$$\ell = \frac{2}{\mu} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2 \cos \theta - 2 \cos \theta_0}} \quad (1.11.11)$$

توجه داشته باشید علامت منفی با جابجا کردن حدود انتگرال گیری حذف شده است.

با استفاده از روابط مثلثاتی زیر:

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \quad \text{and} \quad \cos \theta_0 = 1 - 2 \sin^2 \frac{\theta_0}{2}$$

معادله اخیر به صورت زیر ساده میشود

$$\ell = \frac{1}{\mu} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}$$

$$\sin \frac{\theta_0}{2} = \alpha$$

برای ساده سازی رابطه بالا:

$$\sin \frac{\theta}{2} = \alpha \sin \phi$$

و معرفی متغییر جدید ϕ بصورت مقابل:

$$\frac{1}{2} \cos \frac{\theta}{2} d\theta = \alpha \cos \phi d\phi$$

دیفرانسیل گیری از آن

بدست آوردن $d\theta$ بر حسب $d\phi$

$$d\theta = \frac{2\alpha \cos \phi d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \frac{2\alpha \cos \phi d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}}$$

و حدود انتگرال گیری:

$$\theta = 0 \Rightarrow \phi = 0 \quad \text{and} \quad \theta = \theta_0 \Rightarrow \sin \phi = 1 \Rightarrow \phi = \pi/2.$$

در نتیجه:

$$\begin{aligned} \ell &= \frac{1}{\mu} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} = \frac{1}{\mu} \int_0^{\pi/2} \frac{1}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi}} \frac{2\alpha \cos \phi d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \\ &= \frac{2}{\mu} \int_0^{\pi/2} \frac{1}{\alpha \cos \phi} \frac{\alpha \cos \phi d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \\ \ell &= \frac{2}{\mu} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} = \frac{2K}{\mu} \end{aligned} \quad (1.11.17)$$

where:

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \quad (1.11.18)$$

رابطه ۱، ۱۱، ۱۸ انتگرال بیضوی نوع اول میباشد. مقادیر اینانتگرال به سادگی با نوشتن یک برنامه کامپیوتری قابل حصول است. و رابطه ۱، ۱۱، ۱۷ بصورت زیر قابل باز نویسی است.

$$\ell = \frac{2K}{\mu} = \frac{2K}{\sqrt{P/EI}} \text{ as } \mu^2 = \frac{P}{EI}$$

$$\frac{P}{P_{cr}} = \frac{4K^2}{\pi^2} \quad (1.11.19)$$

$$P = \frac{4K^2}{\ell^2/EI} = \frac{4EIK^2}{\ell^2} \text{ and } P_{cr} = \frac{\pi^2 EI}{\ell^2}$$

اگر تغییرشکلهای تیر ناچیز باشند، آنگاه θ_0 نیز کوچک خواهد بود و $\alpha^2 \sin^2 \phi$ در رابطه مربوط به K ناچیز میشود و مقدار K به $\pi/2$ میل میکند

تغییر مکان وسط دهانه تیر میتواند از رابطه زیر محاسبه گردد:

$$dy = ds \sin \theta.$$

Substituting Eq. (1.11.10) into the above equation yields

$$dy = -\frac{\sin \theta d\theta}{\sqrt{2k\sqrt{\cos \theta} - \cos \theta_0}}$$

Integrating the above equation gives

$$\int_0^{y_m} dy = -\frac{1}{2k} \int_{\theta_0}^0 \frac{\sin \theta d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \text{ or } y_m = \frac{1}{2k} \int_0^{\theta_0} \frac{\sin \theta d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}$$

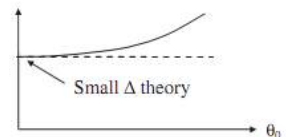
Recall $\sin(\theta/2) = \alpha \sin \phi$ and $d\theta = 2\alpha \cos \phi d\phi / \sqrt{1 - \alpha^2 \sin^2 \phi}$

Hence,

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin \frac{\theta}{2} \sqrt{1 - \sin^2 \frac{\theta}{2}} = 2\alpha \sin \phi \sqrt{1 - \alpha^2 \sin^2 \phi}$$

$$y_m = \frac{1}{2k} \int_0^{\theta_0} \frac{\sin \theta d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}$$

$$= \frac{1}{2k} \int_0^{\pi/2} \frac{2\alpha \sin \phi \sqrt{1 - \alpha^2 \sin^2 \phi} 2\alpha \cos \phi d\phi}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi} \sqrt{1 - \alpha^2 \sin^2 \phi}}$$



$$y_m = \delta = \frac{2\alpha}{k} \int_0^{\pi/2} \sin \phi \, d\phi = \frac{2\alpha}{k} \quad \text{or} \quad \frac{y_m}{\ell} = \frac{2\alpha}{\pi \sqrt{\frac{P}{P_E}}}$$

The distance between the two load points (x-coordinates) can be determined from

$$dx = ds \cos \theta$$

Substituting Eq. (1.11.10) into the above equation yields

$$dx = -\frac{\cos \theta \, d\theta}{\sqrt{2k} \sqrt{\cos \theta - \cos \theta_0}}$$

Integrating (x_m is the x-coordinate at the midheight) the above equation gives

$$\begin{aligned} \int_0^{x_m} dx &= -\frac{1}{\sqrt{2k}} \int_{\theta_0}^0 \frac{\cos \theta \, d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = -\frac{1}{\sqrt{k}} \int_{\theta_0}^0 \frac{\cos \theta \, d\theta}{\sqrt{2 \cos \theta - 2 \cos \theta_0}} \text{ or} \\ x_m &= \frac{1}{2k} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \end{aligned}$$

Recall $\sin(\theta/2) = \alpha \sin \phi$ and $d\theta = 2\alpha \cos \phi \, d\phi / \sqrt{1 - \alpha^2 \sin^2 \phi}$

and $\cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2) = 1 - 2\sin^2(\theta/2) = 1 - 2\alpha^2 \sin^2 \phi$

$$\begin{aligned} x_m &= \frac{1}{2k} \int_0^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \\ &= \frac{1}{2k} \int_0^{\pi/2} \frac{(1 - 2\alpha^2 \sin^2 \phi) 2\alpha \cos \phi \, d\phi}{\sqrt{\alpha^2 - \alpha^2 \sin^2 \phi} \sqrt{1 - \alpha^2 \sin^2 \phi}} \\ &= \frac{1}{k} \int_0^{\pi/2} \frac{(1 - 2\alpha^2 \sin^2 \phi) d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \\ x_0 &= 2x_m = \frac{2}{k} \int_0^{\pi/2} \frac{[2(1 - \alpha^2 \sin^2 \phi) - 1] d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \\ &= \frac{4}{k} \int_0^{\pi/2} \sqrt{1 - \alpha^2 \sin^2 \phi} \, d\phi \\ &\quad - \frac{2}{k} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} = \frac{4}{k} E(\alpha) - \ell \end{aligned}$$

where $E(\alpha)$ is the complete elliptic integral of the second kind

$$\frac{x_0}{\ell} = \frac{4E(\alpha)}{\ell \sqrt{\frac{P}{EI}}} - 1 = \frac{4E(\alpha)}{\pi \sqrt{\frac{P}{P_E}}} - 1$$

The complete elliptic integral of the first kind can be evaluated by an infinite series given by

$$\begin{aligned} K &= \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2 \sin^2 \phi}} \\ &= \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 \alpha^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \alpha^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \alpha^6 + \dots \right] \text{ with } \alpha^2 < 1 \end{aligned}$$

Summing the first four terms of the above infinite series for $\alpha = 0.5$ yields $K = 1.685174$.

Likewise, the complete elliptic integral of the second kind can be evaluated by an infinite series given by

$$\begin{aligned} E &= \int_0^{\pi/2} \sqrt{1 - \alpha^2 \sin^2 \phi} d\phi \\ &= \frac{\pi}{2} \left[1 - \left(\frac{1}{2}\right)^2 \alpha^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{\alpha^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{\alpha^6}{5} - \dots \right] \text{ with } \alpha^2 < 1 \end{aligned}$$

Summing the first four terms of the above infinite series for $\alpha = 0.5$ yields $E = 1.46746$. These two infinite series can be programmed as shown or can

Table 1-1 Load vs. deflection data for large deflection theory

θ_0 / rad	K	E	α	P/P_E	y_m/ℓ	x_0/ℓ
0/0	$\pi/2$	$\pi/2$	.0	1.	.0	1.
20/.349	1.583	1.5588	.174	1.015	.110	0.9700
40/.698	1.620	1.5238	.342	1.063	.211	0.8818
60/1.047	1.686	1.4675	.500	1.152	.296	0.7408
90/1.5708	1.8539	1.3507	.707	1.3929	.3814	0.4572
120/2.0944	2.1564	1.2111	.866	1.8846	.4016	0.1233
150/2.618	2.7677	1.0764	.9659	3.1045	.349	-0.2222
170/2.967	4.4956	1.0040	.999	8.1910	.2222	-0.5533
179.996/ π	12.55264	1.0000	0.9999999999	63.86	.07966	-0.8407

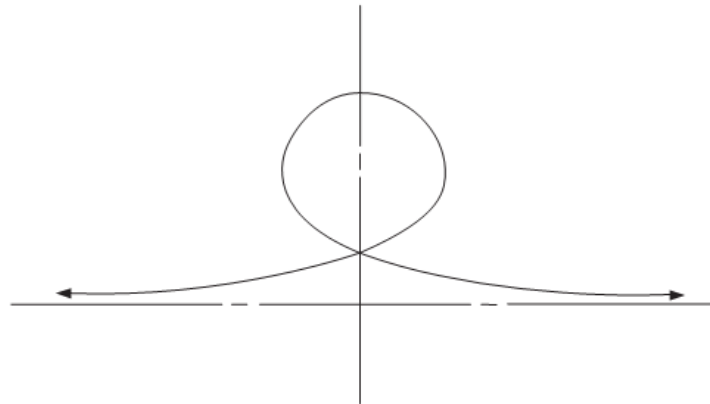


Figure 1-25 Postbuckling shape of wire column

2.2.2 Initial bow or lack of straightness

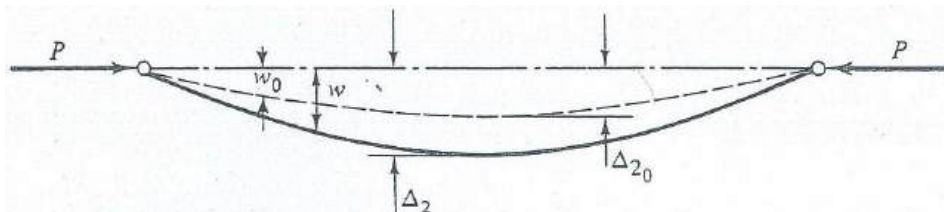
Figure 2.3 shows a uniform elastic pin-ended strut which is slightly bowed when it is not loaded. w and w_0 represent the transverse displacements (from the load axis) when the strut is, respectively, loaded and not loaded.

The bending moment in the loaded strut is $M = Pw$. The bending moment is proportional to the *change* of curvature:

$$M = EI \left(\frac{1}{R} - \frac{1}{R_0} \right) \quad (2.16)$$

where

$$\frac{1}{R} = -\frac{d^2w}{dx^2}, \quad \frac{1}{R_0} = -\frac{d^2w_0}{dx^2}$$



Elimination of M results in the differential equation:

$$\frac{d^2 w}{dx^2} + \mu^2 w = \frac{d^2 w_0}{dx^2}, \quad \text{where} \quad \mu^2 = P/EI. \quad (2.17)$$

Whatever the shape of the unloaded strut, it can be represented as a Fourier series

$$w_0 = \sum a_i \sin \frac{i\pi x}{L}, \quad i = 1, 2, \dots, \infty, \quad (2.18)$$

in which the amplitudes a_i are known or can be found by measurement.

Similarly, the displacements of the loaded strut can be written

$$w = \sum \bar{a}_i \sin \frac{i\pi x}{L}, \quad i = 1, 2, \dots, \infty, \quad (2.19)$$

in which the amplitudes $\bar{a}_i$ are to be found.

Substitution for w and w_0 in Eq. (2.17) gives

$$-\sum \bar{a}_i \frac{i^2 \pi^2}{L^2} \sin \frac{i\pi x}{L} + \mu^2 \sum \bar{a}_i \sin \frac{i\pi x}{L} = -\sum a_i \frac{i^2 \pi^2}{L^2} \sin \frac{i\pi x}{L}. \quad (2.20)$$

The i th term is obtained by omitting the summation signs:

$$-\bar{a}_i \frac{i^2 \pi^2}{L^2} \sin \frac{i\pi x}{L} + \mu^2 \bar{a}_i \sin \frac{i\pi x}{L} = -a_i \frac{i^2 \pi^2}{L^2} \sin \frac{i\pi x}{L}. \quad (2.21)$$

If this is satisfied for all values of i , then Eq. (2.20) is automatically satisfied. It is not difficult to reduce Eq. (2.21) to

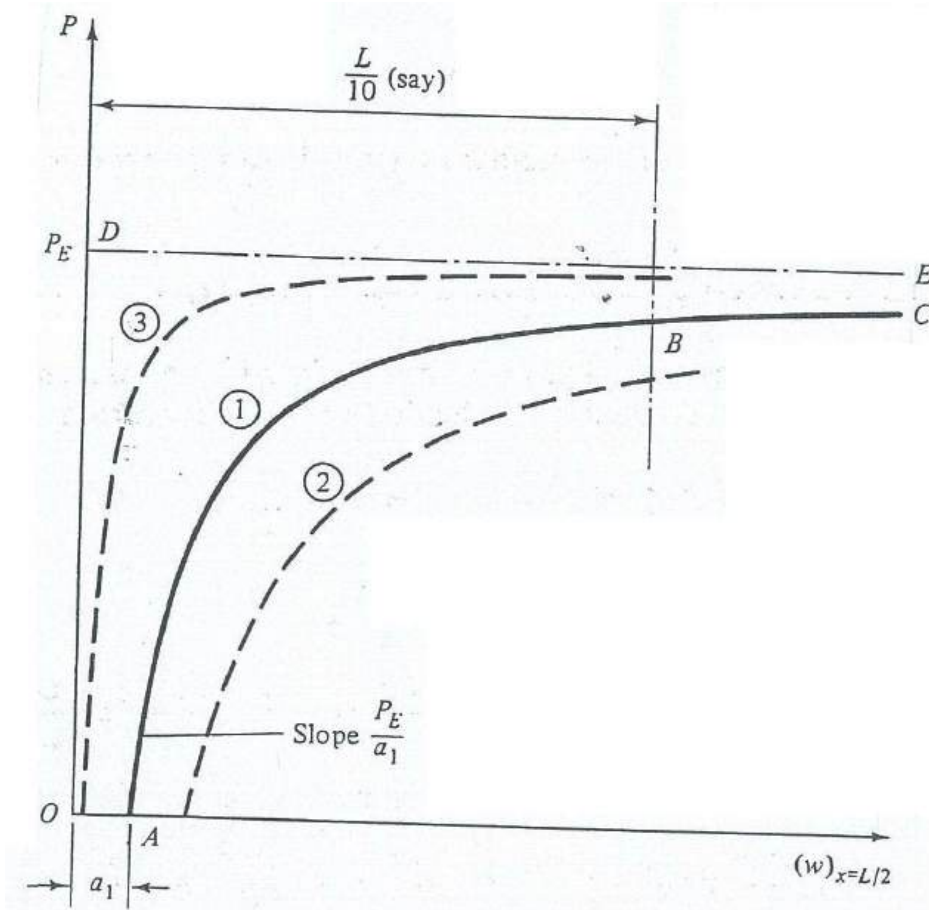
$$\bar{a}_i = \frac{a_i}{1 - (\mu^2 L^2 / i^2 \pi^2)}, \quad \text{or} \quad \bar{a}_i = \frac{a_i}{1 - (P/P_i)}, \quad \text{where} \quad P_i = \frac{i^2 \pi^2 EI}{L^2}. \quad (2.22)$$

The effect of the thrust P is to increase the amplitude of the i th term of the original Fourier series by an *amplification factor*, $1/[1 - (P/P_i)]$, which becomes infinitely large as $P \rightarrow P_i$.

Provided the shape of the unloaded strut is known and can be broken down into its Fourier components (a_i), then the Fourier components for the loaded strut ($\bar{a}_i$) can be found from Eq. (2.22); the total deformation of the loaded strut can be found from Eq. (2.19).

Suppose the load is increased steadily from zero. As it approaches the first critical load (P_1 , or P_E), the amplitude of the first mode becomes very large—larger than all the others, which can be neglected in consequence. It follows that for loads close to the Euler load, the displacement can be written

$$w = \frac{a_1}{1 - (P/P_E)} \sin \frac{\pi x}{L}. \quad (2.23)$$



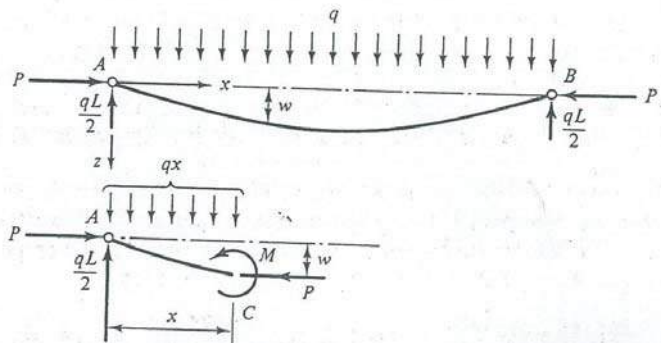
Transverse Load:

Uniformly Distributed Transverse load (pin ended):

$$M = Pw + \frac{qL}{2}x - \frac{qx^2}{2}$$

$$\frac{d^2w}{dx^2} + \mu^2w = \frac{q}{EI} \left(\frac{x^2}{2} - \frac{xL}{2} \right)$$

$$w = A \sin \mu x + B \cos \mu x + \frac{q}{P} \left(\frac{x^2}{2} - \frac{xL}{2} - \frac{1}{\mu^2} \right)$$



The boundary conditions

$w = 0$ at $x = 0$ and $x = L$,

$$B = \frac{q}{P\mu^2}$$

$$A = \frac{q}{P\mu^2} \left(\frac{1 - \cos \mu L}{\sin \mu L} \right) = \frac{q}{P\mu^2} \tan \frac{\mu L}{2}$$

$$w = \frac{q}{P\mu^2} \left(\tan \frac{\mu L}{2} \sin \mu x + \cos \mu x - 1 \right) + \frac{q}{2P} (x^2 - xL)$$

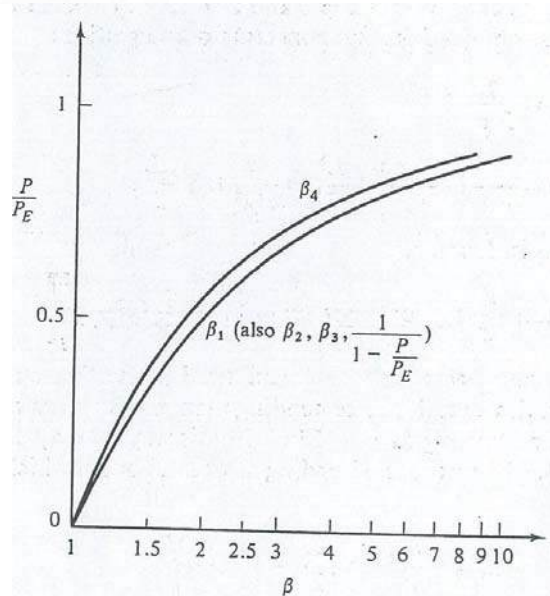
$$\text{at } x = L/2: w_{\max} = \frac{q}{P\mu^2} \left(\sec \frac{\mu L}{2} - 1 \right) - \frac{qL^2}{8P}, \quad M_{\max} = \frac{qEI}{P} \left(\sec \frac{\mu L}{2} - 1 \right)$$

Uniformly Distributed Transverse load (pin ended): continue

$$w_{\max} = \frac{5}{384} \frac{qL^4}{EI} \beta_1, \quad M_{\max} = \frac{qL^2}{8} \beta_2,$$

$$\beta_1 = \frac{12}{5\phi^4} \{2(\sec \phi - 1) - \phi^2\}, \quad \beta_2 = \frac{2}{\phi^2} (\sec \phi - 1),$$

$$\phi = \frac{\mu L}{2} = \frac{L}{2} \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \sqrt{\frac{P}{P_E}}.$$

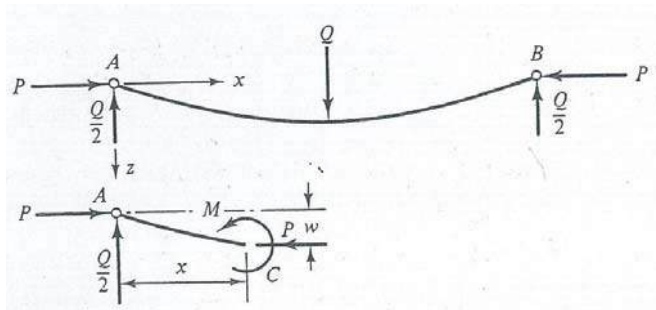


Central point load (pin ended):

$$M = +Pw + \frac{Q}{2}x.$$

$$\frac{d^2w}{dx^2} + \mu^2 w = -\frac{Q}{2EI}x,$$

$$w = A \sin \mu x + B \cos \mu x - \frac{Q}{2P}x.$$



The boundary conditions ($w=0$ at $x=0$ and $w'=0$ at $x=L/2$)

$$B = 0; \quad A = \frac{Q}{2P\mu} \sec \frac{\mu L}{2}.$$

$$w = \frac{Q}{2P} \left(\frac{\sin \mu x}{\mu \cos \frac{\mu L}{2}} - x \right), \quad 0 \leq x \leq L/2.$$

$$\text{at } x = L/2 \quad w_{\max} = \frac{QL}{4P} \left(\frac{\tan \frac{\mu L}{2}}{\frac{\mu L}{2}} - 1 \right), \quad M_{\max} = \frac{QL}{4} \left(\frac{\tan \frac{\mu L}{2}}{\frac{\mu L}{2}} \right).$$

$$w_{\max} = \frac{QL^3}{48EI} \beta_3, \quad M_{\max} = \frac{QL}{4} \beta_4.$$

$$\beta_3 = \frac{3}{\phi^3} (\tan \phi - \phi), \quad \beta_4 = \frac{\tan \phi}{\phi},$$

$$\phi = \frac{\mu L}{2} = \frac{L}{2} \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \sqrt{\frac{P}{P_E}}.$$

End-couples: stiffness and carry-over factors

$$S = -(M_a + M_b)/L.$$

$$M = +Pw + Sx + M_a = -EI \frac{d^2 w}{dx^2}$$

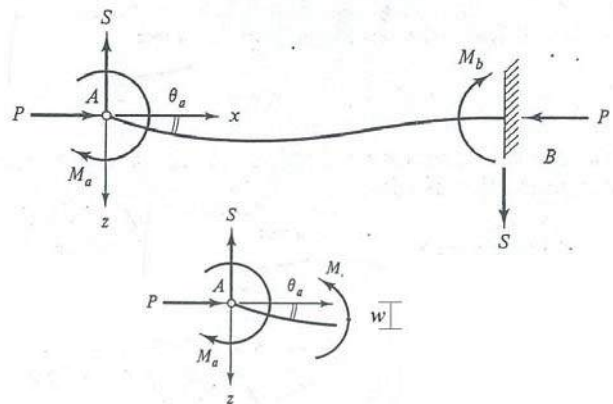
$$\frac{d^2 w}{dx^2} + \mu^2 w = -\frac{Sx}{EI} - \frac{M_a}{EI}$$

$$w = A \sin \mu x + B \cos \mu x - \frac{Sx}{P} - \frac{M_a}{P}.$$

The boundary conditions are:

$$w = 0 \quad \text{and} \quad w' = \theta_a \quad \text{at} \quad x = 0$$

$$w = 0 \quad \text{and} \quad w' = 0 \quad \text{at} \quad x = L.$$



$$M_a = sk\theta_a, \quad M_b = cM_a$$

$$s = \frac{(\mu L/2)(1 - \mu L \cot \mu L)}{\tan(\mu L/2) - (\mu L/2)}, \quad c = \frac{\mu L - \sin \mu L}{\sin \mu L - \mu L \cos \mu L},$$

$$k = \frac{EI}{L}.$$

Stability functions:

$$q = \frac{PL^2}{\pi^2 EI}$$

Tensile Axial Force, ($q < 0$):

$$s = \frac{\psi^2 \cosh \psi - \psi \sinh \psi}{2 - 2 \cosh \psi - \psi \sinh \psi}$$

$$SC = \frac{\psi \sinh \psi - \psi^2}{2 - 2 \cosh \psi - \psi \sinh \psi}$$

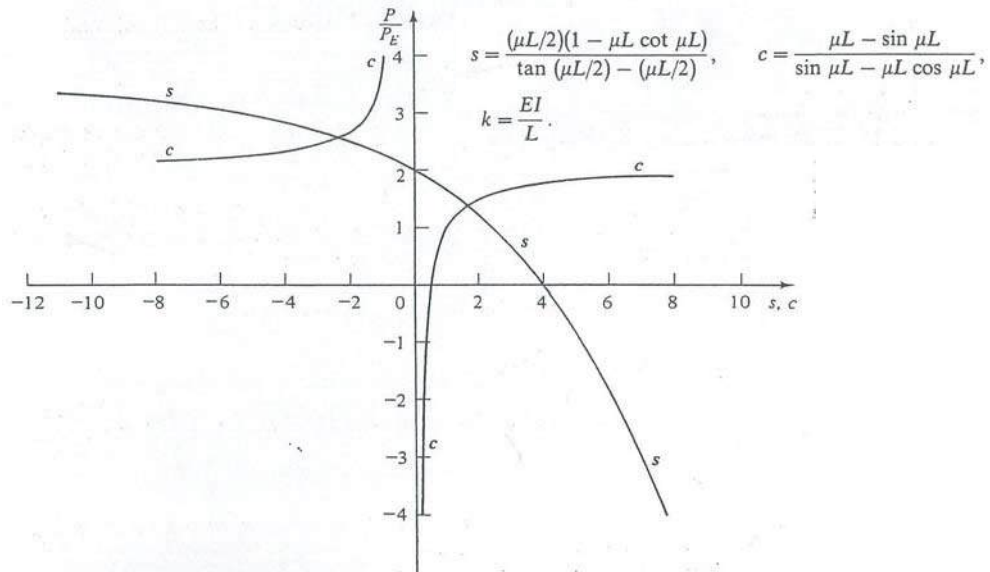
in which $\psi^2 = -\pi^2 q \dots\dots$

Compressive Axial Force, ($q > 0$):

$$S = \frac{\phi \sin \phi - \phi^2 \cos \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$SC = \frac{\phi^2 - \phi \sin \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

in which $\phi^2 = \pi^2 q \dots\dots$

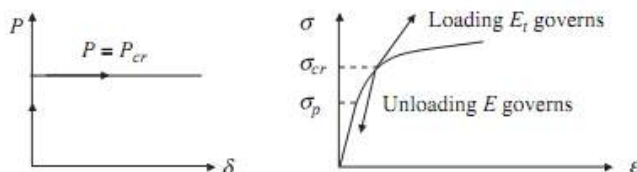
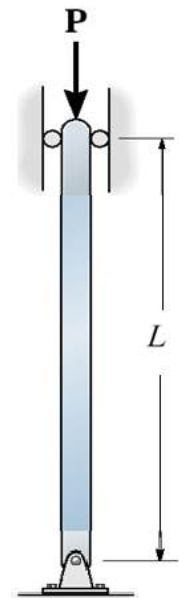


1.13. INELASTIC BUCKLING OF STRAIGHT COLUMN

1.13.1. Double-Modulus (Reduced Modulus) Theory

Assumptions

- 1) Small displacement theory holds.
- 2) Plane sections remain plane. This assumption is called Bernoulli, or Euler, or Navier hypothesis.
- 3) The relationship between the stress and strain in any longitudinal fiber is given by the stress-strain diagram of the material (compression and tension, the same relationship).
- 4) The column section is at least singly symmetric, and the plane of bending is the plane of symmetry.
- 5) The axial load remains constant as the member moves from the straight to the deformed position.



$$\frac{1}{\rho} = \frac{d^2 w}{dx^2} = \frac{d\theta}{dx}$$

$$z_1 d\theta = \varepsilon_1 dx \quad \& \quad z_2 d\theta = \varepsilon_2 dx$$

$$z_1 d\theta / dx = \varepsilon_1 \quad \& \quad z_2 d\theta / dx = \varepsilon_2$$

$$\varepsilon_1 = z_1 w'' \quad \varepsilon_2 = z_2 w''$$

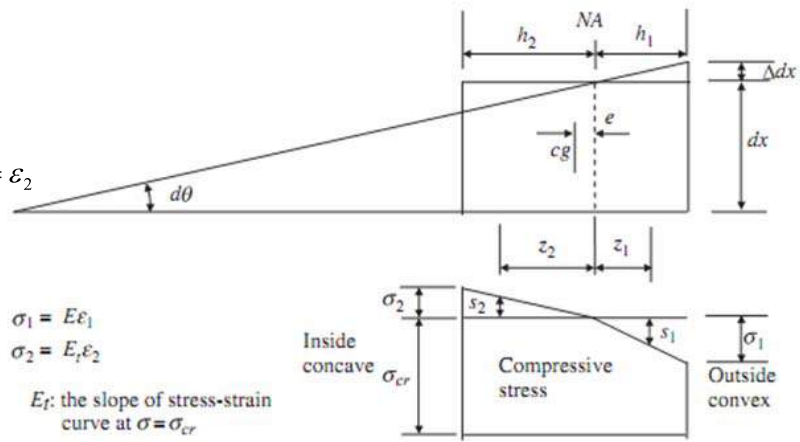
$$\sigma_1 = E\varepsilon_1 = Ez_1 w''$$

$$\sigma_2 = E_t \varepsilon_2 = E_t z_2 w''$$

$$\sigma_1 = E\varepsilon_1$$

$$\sigma_2 = E_t \varepsilon_2$$

E_t : the slope of stress-strain curve at $\sigma = \sigma_{cr}$



$$\int_{A_1} \sigma_1 dA_1 + \int_{A_2} \sigma_2 dA_2 = \int_{A_1} Ez_1 w'' dA_1 + \int_{A_2} E_t z_2 w'' dA_2 = 0$$

$$w'' (E \int_{A_1} z_1 dA_1 + E_t \int_{A_2} z_2 dA_2) = 0 \Rightarrow EQ_1 + E_t Q_2 = 0$$

$$\int_{A_1} \sigma_1 z_1 dA_1 + \int_{A_2} \sigma_2 z_2 dA_2 = \int_{A_1} Ez_1^2 w'' dA_1 + \int_{A_2} E_t z_2^2 w'' dA_2 = -Pw$$

$$w'' (E \int_{A_1} z_1^2 dA_1 + E_t \int_{A_2} z_2^2 dA_2) = -Pw \Rightarrow (EI_1 + E_t I_2) w'' + Pw = 0$$

$$E_r I = EI_1 + E_t I_2 \Rightarrow E_r = \frac{EI_1 + E_t I_2}{I}$$

Tangent-Modulus Theory

Assumptions

The assumptions are the same as those used in the double-modulus theory, except assumption 5. The axial load increases during the transition from the straight to slightly bent position, such that the increase in average stress in compression is greater than the decrease in stress due to bending at the extreme fiber on the convex side. The compressive stress increases at all points; the tangent modulus governs the entire cross section.

If the load increment is assumed to be negligibly small such that:

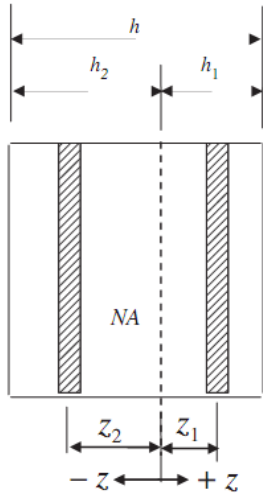
$$\Delta P \ll \ll P$$

$$E_t I y'' + P y = 0$$

An axially loaded, simply supported column is made of structural steel with the following mechanical properties:

$E = 30000 \text{ ksi}$; $\sigma_P = 28.0 \text{ ksi}$ $\sigma_y = 36 \text{ ksi}$; and tangent moduli given in Table 1-2.

Tangent moduli measured										
$\sigma_t \text{ or } \sigma_r (\text{ksi})$	28	29	30	31	32	33	34	35	35.5	36
$\tau = E_t/E$	1.00	0.98	0.96	0.93	0.88	0.77	0.55	0.31	0.16	0.00



Locations of NA:

$$EQ_1 + E_t Q_2 = 0 \quad h_1 + h_2 = h$$

$$Q_1 = \int_0^{h_1} z_1 dA \quad Q_2 = \int_0^{h_2} z_2 dA$$

$$Q_1 = \int_0^{h_1} z_1 h dz_1 = \frac{h}{2} z_1^2 \Big|_0^{h_1} = \frac{h h_1^2}{2}$$

$$Q_2 = -(h h_2^2 / 2)$$

$$E b h_1^2 / 2 = E_t b h_2^2 / 2 = E_t b (h - h_1)^2 / 2$$

$$h_1 = h \sqrt{E_t} / (\sqrt{E} + \sqrt{E_t}) \quad \& \quad h_2 = h \sqrt{E} / (\sqrt{E} + \sqrt{E_t})$$

$$I_1 = \frac{b h_1^3}{3} = \frac{b h^3}{3} \frac{E_t \sqrt{E_t}}{(\sqrt{E} + \sqrt{E_t})^3} \quad \& \quad I_2 = \frac{b h_2^3}{3} = \frac{b h^3}{3} \frac{E \sqrt{E}}{(\sqrt{E} + \sqrt{E_t})^3}$$

$$E_r = \frac{E I_1 + E_t I_2}{I} = \frac{4 E E_t}{(\sqrt{E} + \sqrt{E_t})^2}$$

E_t/E	h_1/h	I_1/I	I_2/I	(1) × (4)	$\frac{E_r/E}{[(3) + (5)]}$	$\sigma, \sigma_t, \sigma_r$	$\sigma/\tau [(7)/(1)]$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.00	.5000	.5000	.5000	.5000	1.0000	28.0	28.00
0.98	.4975	.4925	.5076	.4975	0.9899	29.0	29.60
0.96	.4950	.4848	.5155	.4948	0.9797	30.0	31.25
0.93	.4910	.4733	.5277	.4908	0.9640	31.0	33.33
0.88	.4840	.4536	.5495	.4835	0.9371	32.0	36.36
0.77	.4674	.4084	.6044	.4654	0.8738	33.0	42.86
0.55	.4258	.3088	.7572	.4165	0.7253	34.0	61.82
0.31	.3576	.1830	1.0602	.3287	0.5116	35.0	112.90
0.16	.2857	.0933	1.4577	.2332	0.3265	35.5	221.88
0.00	.0000	.0000	4.0000	.0000	0.0000	36.0	∞

- 2) The value of τ_r and ℓ/r for $P/A = 28, 30, 32, 34, 35, 35.5$ ksi using the double-modulus theory and assuming that the cross section of the column is a square of side h .
- 3) The critical average stress P/A for $\ell/r = 20, 40, 60, 80, 100, 120, 140, 160, 180$, and 200 using the tangent-modulus theory in the inelastic range.

ℓ/r	σ_r	ℓ/r	λ_c	$F_{cr \text{ AISC}}$	σ_t/τ	σ_t	Remarks
102.83	28.0	200	2.21	6.49	7.402	7.402	} elastic
100.53	29.0	180	1.98	8.01	9.138	9.138	
98.33	30.0	160	1.76	10.14	11.566	11.566	
95.96	31.0	140	1.54	13.25	15.107	15.107	
93.12	32.0	120	1.32	18.03	20.562	20.562	} σ_t , from graph
88.54	33.0	100	1.10	21.64	29.609	29.000	
79.47	34.0	80	0.88	25.99	46.264	33.200	
65.79	35.0	60	0.66	29.97	82.247	34.200	
52.18	35.5	40	0.44	33.18	185.055	35.300	
0.00	36.0	20	0.22	35.27	740.220	35.990	
		0	0.00	36.00	∞	36.000	

Example : Consider Simply supported column with rectangular cross section and *Stress–Strain relation* :

$$\sigma = \frac{\sigma_y}{5} \left(4 + \tanh\left(\frac{5E\varepsilon}{\sigma_y} - 4\right) \right) \quad \text{for } \sigma > \sigma_p \quad \& \quad \sigma = E\varepsilon \quad \text{for } \sigma \leq \sigma_p \quad \text{where } \sigma_p = 0.8\sigma_y \quad \text{Find : } \sigma - L/r \text{ relation for } E, E_r \& E_t.$$

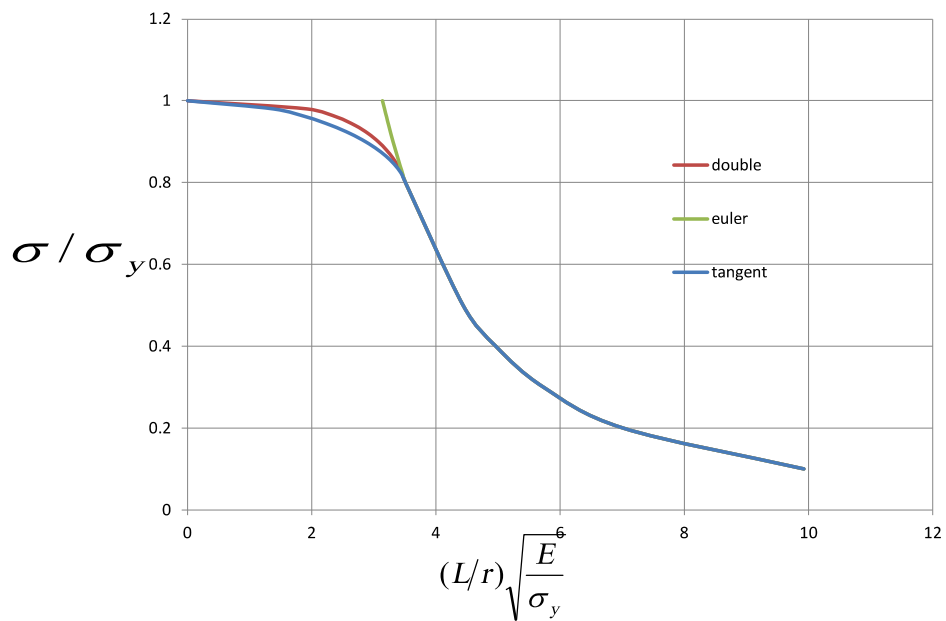
$$\text{Tangent modulus : } E_t = \frac{d\sigma}{d\varepsilon} = E \left(1 - \tanh^2\left(\frac{5E\varepsilon}{\sigma_y} - 4\right) \right) = E \left(1 - \left(\frac{5\sigma}{\sigma_y} - 4\right)^2 \right)$$

$$\text{Reduced modulus : } E_r = \frac{4EE_t}{(\sqrt{E} + \sqrt{E_t})^2}$$

$$(\sigma_{cr})_E = \frac{\pi^2 E}{(L/r)_E^2} \Rightarrow (L/r)_E = \pi \sqrt{\frac{E}{(\sigma_{cr})_E}} \quad \& \quad (\sigma_{cr})_r = \frac{\pi^2 E_r}{(L/r)_r^2} \Rightarrow (L/r)_r = \pi \sqrt{\frac{E}{(\sigma_{cr})_r}}$$

$$(\sigma_{cr})_t = \frac{\pi^2 E_t}{(L/r)_t^2} \Rightarrow (L/r)_t = \pi \sqrt{\frac{E_t}{(\sigma_{cr})_t}}$$

σ / σ_y	E/E	Et/E	Er/E	$(L/r)_E / \sqrt{\frac{E}{\sigma_y}}$	$(L/r)_t / \sqrt{\frac{E_t}{\sigma_y}}$	$(L/r)_r / \sqrt{\frac{E_r}{\sigma_y}}$
0-0.8	1	1	1	3.141 $\sqrt{\sigma_y/\sigma}$	3.141 $\sqrt{\sigma_y/\sigma}$	3.141 $\sqrt{\sigma_y/\sigma}$
0.82	1	0.99	0.995	3.469	3.452	3.461
0.84	1	0.96	0.980	3.428	3.359	3.393
0.86	1	0.91	0.953	3.388	3.232	3.308
0.88	1	0.84	0.915	3.349	3.069	3.203
0.9	1	0.75	0.862	3.312	2.868	3.074
0.92	1	0.64	0.790	3.275	2.620	2.911
0.94	1	0.51	0.694	3.240	2.314	2.700
0.96	1	0.36	0.563	3.206	1.924	2.405
0.98	1	0.19	0.369	3.173	1.383	1.927
1	1	0	0.000	3.142	0.000	0.000



$$F_a = \frac{1}{F.S.} \left[1 - \frac{1}{2} \left(\frac{\lambda}{C_c} \right)^2 \right] F_y$$

$$C_c = \text{آن کمتر از } \left(\lambda = \frac{KL}{r} \right)_{\max}$$

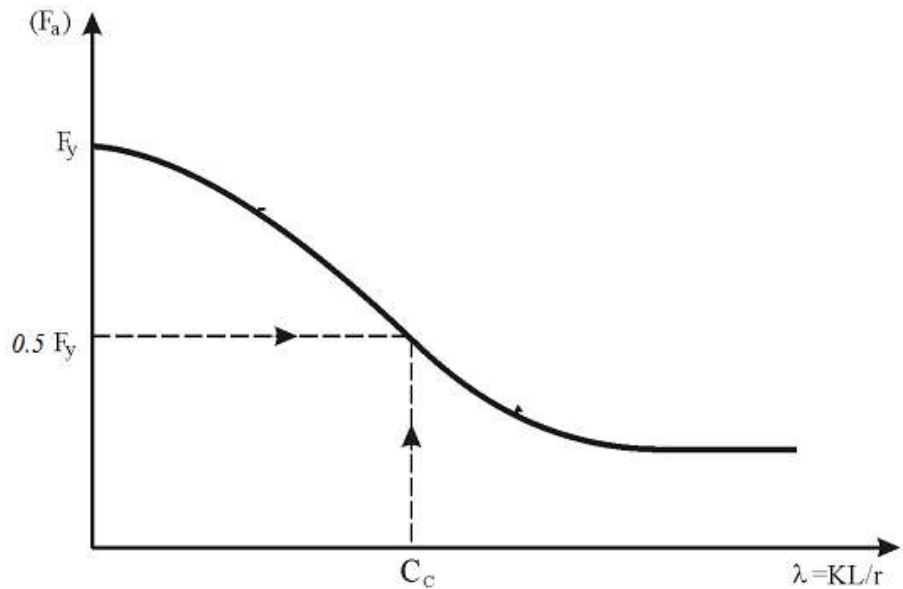
$$F.S. = 1/67 + 0/375 \left(\frac{\lambda}{C_c} \right) - 0/125 \left(\frac{\lambda}{C_c} \right)^2$$

$$F_a = \frac{12 \pi^2 E}{23(\lambda)^2} = \frac{105 \times 10^5}{\lambda^2}$$

$$C_c = \text{بزرگتر از } \left[\lambda = \frac{KL}{r} \right]_{\max}$$

$$C_c = \sqrt{\frac{2 \pi^2 E}{F_y}} = \frac{6440}{\sqrt{F_y}}$$

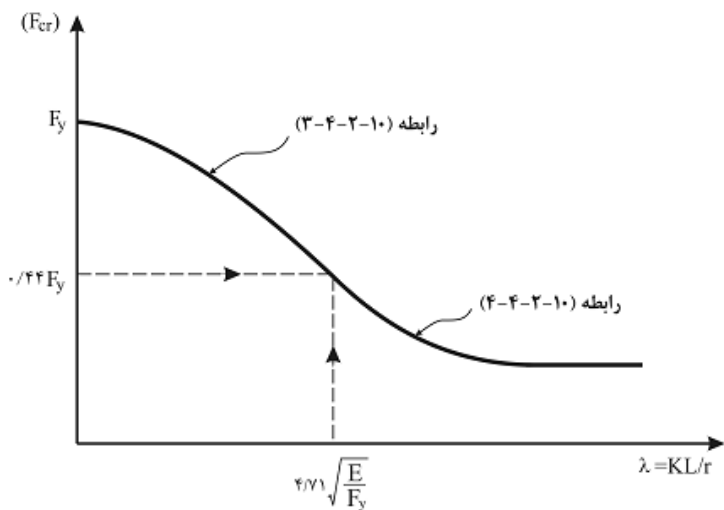
**AISC
ASD**



AISC LRFD

$$\lambda \leq \phi / \gamma_1 \sqrt{\frac{E}{F_y}}$$

$$F_{cr} = \left[0/658 \frac{F_y}{F_c} \right] F_y$$

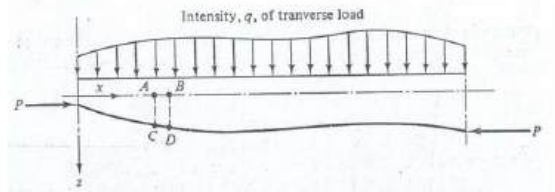


$$\lambda > \phi / \gamma_1 \sqrt{\frac{E}{F_y}}$$

$$F_{cr} = 0/45 F_c$$

APPROXIMATE METHODS: SUBSTITUTION OF A SYSTEM WITH A FINITE NUMBER OF DEGREES OF FREEDOM FOR ONE WITH AN INFINITE NUMBER OF APPROXIMATIONS USING CONTINUOUS DISPLACEMENT FUNCTIONS

Rayleigh-Ritz method: approximations using one degree of freedom



$$V = U + V_p = \int_0^L \frac{EI}{2} w''^2 dx - \frac{P}{2} \int_0^L w'^2 dx - \int_0^L q w dx \quad w = a\phi(x)$$

$$V = U + V_p = a^2 \int_0^L \frac{EI}{2} \phi''^2 dx - \frac{P}{2} a^2 \int_0^L \phi'^2 dx - a \int_0^L q \phi dx$$

$$\frac{dV}{da} = a \left(\int_0^L EI \phi''^2 dx - P \int_0^L \phi'^2 dx \right) - \int_0^L q \phi dx = 0$$

$$\text{Trivial Sol.: } a = \frac{\int_0^L q \phi dx}{\int_0^L EI \phi''^2 dx - P \int_0^L \phi'^2 dx} \quad \text{Nontrivial Sol.: } P_{cr} = \frac{\int_0^L EI \phi''^2 dx}{\int_0^L \phi'^2 dx}$$

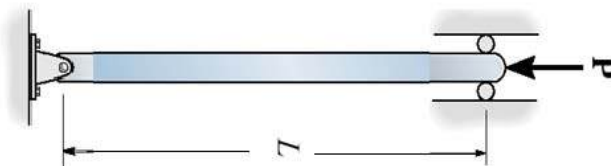
Example No. 1

Suppose that the strut is pin-ended and has a uniform cross-section (EI constant). A suitable deflected shape might be the parabola,

$$\phi(x) = xL - x^2.$$

Substitution in Eq. (1.101) yields the following result for the critical load:

$$P_{cr} = \frac{\int_0^L EI \phi''^2 dx}{\int_0^L \phi'^2 dx} \quad P = \frac{EI \int_0^L (-2)^2 dx}{\int_0^L (L - 2x)^2 dx} = \frac{12EI}{L^2}.$$



Example No. 2

A better approximation is possible if the parabola is replaced by the deflection curve for a simply supported beam with a uniformly distributed load:

$$\phi(x) = x^4 - 2x^3L + xL^3. \quad (1.104)$$

This satisfies all the boundary conditions for a pin-ended strut ($\phi = \phi'' = 0$ at $x = 0, L$) and it is easy to show that when it is used in Eq. (1.101), the critical load is

$$P = \frac{9.88EI}{L^2}. \quad (1.105)$$

This is almost indistinguishable from the correct value. Clearly, *the best approximations are obtained when $\phi(x)$ is chosen to satisfy all the boundary conditions*

Example No. 3

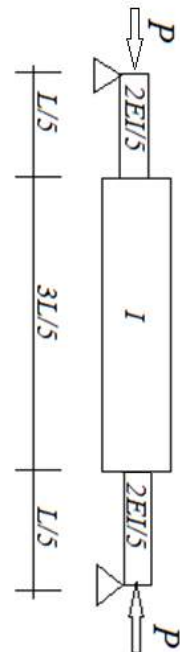
Suppose that a sine curve is chosen as the displacement function:

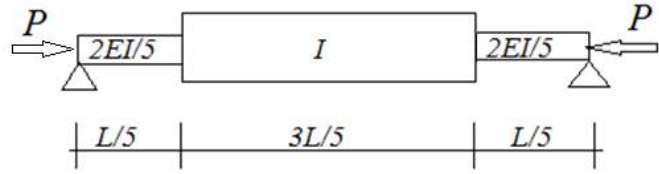
$$\phi(x) = \sin \frac{\pi x}{L}; \quad \phi'(x) = \frac{\pi}{L} \cos \frac{\pi x}{L}; \quad \phi''(x) = -\frac{\pi^2}{L^2} \sin \frac{\pi x}{L}. \quad (1.106)$$

Equation (1.101) gives the critical load as

$$P = \frac{EI \int_0^L \frac{\pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx}{\int_0^L \frac{\pi^2}{L^2} \cos^2 \frac{\pi x}{L} dx} = \frac{EI \frac{\pi^4}{L^4} \frac{L}{2}}{\frac{\pi^2}{L^2} \frac{L}{2}} = \frac{\pi^2 EI}{L^2}. \quad (1.107)$$

$$\begin{aligned} \int_0^L (\phi')^2 dx &= \int_0^L \frac{\pi^2}{L^2} \cos^2 \frac{\pi x}{L} dx = \frac{\pi^2}{L^2} \frac{L}{2}, \\ \int_0^L EI(\phi'')^2 dx &= 2 \int_0^{L/5} (0.4EI) \frac{\pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx \\ &\quad + 2 \int_{L/5}^{L/2} (EI) \frac{\pi^4}{L^4} \sin^2 \frac{\pi x}{L} dx \\ &= 0.8EI \frac{\pi^4}{L^4} \frac{L}{2} \left(\frac{1}{5} - \frac{1}{2\pi} \sin \frac{2\pi}{5} \right) \\ &\quad + 2EI \frac{\pi^4}{L^4} \frac{L}{2} \left(\frac{L}{2} - \frac{L}{2\pi} \sin \pi - \frac{L}{5} + \frac{L}{2\pi} \sin \frac{2\pi}{5} \right) \\ &= EI \frac{\pi^4}{L^4} \frac{L}{2} (0.8 \times 0.04864 + 2 \times 0.4514). \end{aligned}$$





$$\phi(x) = \sin \frac{\pi x}{L}$$

$$P_{cr} = \frac{\int_0^L EI(x) \phi''^2 dx}{\int_0^L \phi'^2 dx} = \frac{\frac{2}{5} \int_0^{L/5} \left(\sin \frac{\pi x}{L}\right)^2 dx + \int_{L/5}^{4L/5} \left(\sin \frac{\pi x}{L}\right)^2 dx + \frac{2}{5} \int_{4L/5}^L \left(\sin \frac{\pi x}{L}\right)^2 dx}{\int_0^L \left(\cos \frac{\pi x}{L}\right)^2 dx} \frac{\pi^2}{L^2} EI = 9.2936 \frac{EI}{L^2}$$

$$\text{Exact Solution : } P_{cr} = 8.51 \frac{EI}{L^2} \quad \text{Err : 9.2\%}$$

Rayleigh-Ritz method†: approximations using two or more degrees of freedom

$$V = U + V_p = \int_0^L \frac{EI}{2} w''^2 dx - \frac{P}{2} \int_0^L w'^2 dx - \int_0^L q w dx \quad w = \sum a_i \phi_i(x)$$

$$V = U + V_p = \int_0^L \frac{EI}{2} \left(\sum a_i \phi_i''(x)\right)^2 dx - \frac{P}{2} \int_0^L \left(\sum a_i \phi_i'(x)\right)^2 dx - \int_0^L q \sum a_i \phi_i dx$$

$$\frac{\partial V}{\partial a_j} = \int_0^L EI \left(\sum a_i \phi_i''(x)\right) \phi_j''(x) dx - P \int_0^L \left(\sum a_i \phi_i'(x)\right) \phi_j'(x) dx - \int_0^L q \phi_j dx = 0 \quad j = 1, 2, \dots$$

$$\text{Or: } \begin{bmatrix} \int_0^L EI (\phi_1'')^2 dx & \int_0^L EI \phi_1'' \phi_2'' dx & \dots & \int_0^L EI \phi_1'' \phi_n'' dx \\ \int_0^L EI \phi_2'' \phi_1'' dx & \int_0^L EI (\phi_2'')^2 dx & \dots & \int_0^L EI \phi_2'' \phi_n'' dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^L EI \phi_n'' \phi_1'' dx & \int_0^L EI \phi_n'' \phi_2'' dx & \dots & \int_0^L EI (\phi_n'')^2 dx \end{bmatrix} - P \begin{bmatrix} \int_0^L (\phi_1')^2 dx & \int_0^L \phi_1' \phi_2' dx & \dots & \int_0^L \phi_1' \phi_n' dx \\ \int_0^L \phi_2' \phi_1' dx & \int_0^L (\phi_2')^2 dx & \dots & \int_0^L \phi_2' \phi_n' dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^L \phi_n' \phi_1' dx & \int_0^L \phi_n' \phi_2' dx & \dots & \int_0^L (\phi_n')^2 dx \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \int_0^L q \phi_1 dx \\ \int_0^L q \phi_2 dx \\ \vdots \\ \int_0^L q \phi_n dx \end{Bmatrix}$$

$$\text{Or: } ([U] - P[V])\{a\} = \{q\} \quad U_{ij} = \int_0^L EI \phi_i'' \phi_j'' dx \quad V_{ij} = \int_0^L \phi_i' \phi_j' dx$$

$$\phi_1(x) = (x^4 - 2Lx^3 + L^3x)/L^4,$$

$$\phi_2(x) = (4x^3L - 3L^3x)/L^4 \quad \left(0 \leq x \leq \frac{L}{2}\right).$$

$$\begin{bmatrix} (24 - 136p) & (-75 + 427p) \\ (-75 + 427p) & (240 - 1344p) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad p = PL^2/56EI.$$

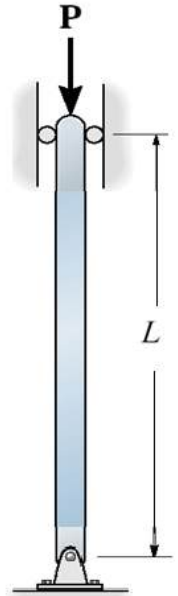
determinant of the square matrix vanishes, viz:

$$455p^2 - 846p + 135 = 0. \quad p = 0.17629, 1.68305,$$

$$P = \frac{9.85EI}{L^2} \quad \text{and} \quad \frac{94EI}{L^2}.$$

$$\frac{a_2}{a_1} = \frac{24 - 136p}{75 - 427p} \quad \text{or} \quad \frac{75 - 427p}{240 - 1344p}.$$

$$w = a_1(\phi_1(x) - 0.0895\phi_2(x)).$$



$$w = a_1\phi_1 + a_2\phi_2$$

$$\phi_1(x) = \sin \frac{\pi x}{L} \quad \phi_2(x) = \sin \frac{3\pi x}{L}$$

$$\phi_1'(x) = \frac{\pi}{L} \cos \frac{\pi x}{L} \quad \phi_2'(x) = \frac{3\pi}{L} \cos \frac{3\pi x}{L}$$

$$\phi_1''(x) = -\frac{\pi^2}{L^2} \sin \frac{\pi x}{L} \quad \phi_2''(x) = -\frac{9\pi^2}{L^2} \sin \frac{3\pi x}{L}$$

$$U_{ij} = \int_0^L EI(x)\phi_i''\phi_j''dx \quad V_{ij} = \int_0^L \phi_i'\phi_j'dx$$

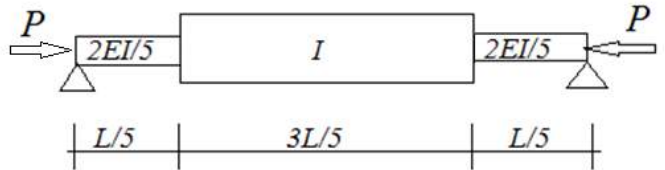
$$U_{11} = 45.862 \frac{EI}{L^3} \quad U_{12} = U_{21} = 24.451 \frac{EI}{L^3} \quad U_{22} = 2143.55 \frac{EI}{L^3}$$

$$V_{P11} = \frac{4.935}{L} \quad V_{P12} = V_{P21} = 0 \quad U_{22} = \frac{44.41}{L}$$

$$\begin{bmatrix} 45.862 \frac{EI}{L^3} - \frac{4.935P}{L} & 24.451 \frac{EI}{L^3} \\ 24.451 \frac{EI}{L^3} & 2143.55 \frac{EI}{L^3} - \frac{44.41P}{L} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$P_{cr1} = 9.2233 \frac{EI}{L^2}$$

$$\text{Exact Solution : } P_{cr} = 8.51 \frac{EI}{L^2} \quad \text{Err : 8.4\%}$$

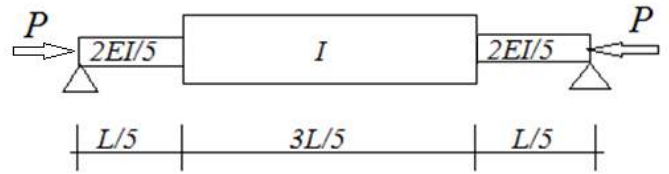


$$w = a_1 \phi_1 + a_2 \phi_2$$

$$\phi_1(x) = \sin \frac{\pi x}{L} \quad \phi_2(x) = \sin \frac{2\pi x}{L}$$

$$\phi_1'(x) = \frac{\pi}{L} \cos \frac{\pi x}{L} \quad \phi_2'(x) = \frac{2\pi}{L} \cos \frac{2\pi x}{L}$$

$$\phi_1''(x) = -\frac{\pi^2}{L^2} \sin \frac{\pi x}{L} \quad \phi_2''(x) = -\frac{4\pi^2}{L^2} \sin \frac{2\pi x}{L}$$



$$U_{ij} = \int_0^L EI(x) \phi_i'' \phi_j'' dx \quad V_{ij} = \int_0^L \phi_i' \phi_j' dx$$

$$U_{11} = 45.862 \frac{EI}{L^3} \quad U_{12} = U_{21} = 0 \quad U_{22} = 635.987 \frac{EI}{L^3}$$

$$V_{p11} = \frac{4.935}{L} \quad V_{p12} = V_{p21} = 0 \quad U_{22} = \frac{19.96}{L}$$

$$\begin{bmatrix} 45.862 \frac{EI}{L^3} - \frac{4.935P}{L} & 0 \\ 0 & 635.987 \frac{EI}{L^3} - \frac{19.96P}{L} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

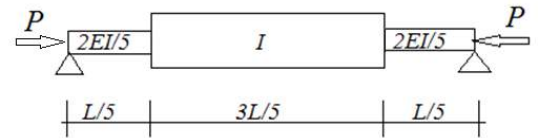
$$P_{cr1} = 9.2936 \frac{EI}{L^2} \quad P_{cr2} = 32.219 \frac{EI}{L^2}$$

$$w = a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3$$

$$\phi_1(x) = \sin \frac{\pi x}{L} \quad \phi_2(x) = \sin \frac{3\pi x}{L} \quad \phi_3(x) = \sin \frac{5\pi x}{L}$$

$$\phi_1'(x) = \frac{\pi}{L} \cos \frac{\pi x}{L} \quad \phi_2'(x) = \frac{3\pi}{L} \cos \frac{3\pi x}{L} \quad \phi_3'(x) = \frac{5\pi}{L} \cos \frac{5\pi x}{L}$$

$$\phi_1''(x) = -\frac{\pi^2}{L^2} \sin \frac{\pi x}{L} \quad \phi_2''(x) = -\frac{9\pi^2}{L^2} \sin \frac{3\pi x}{L} \quad \phi_3''(x) = -\frac{25\pi^2}{L^2} \sin \frac{5\pi x}{L}$$



$$\begin{bmatrix} 45.862 \frac{EI}{L^3} - \frac{4.935P}{L} & -24.451 \frac{EI}{L^3} & -113.906 \frac{EI}{L^3} \\ & 2143.55 \frac{EI}{L^3} - \frac{44.41P}{L} & -2488.11 \frac{EI}{L^3} \\ \text{SYM} & & 23134.689 \frac{EI}{L^3} - \frac{123.37P}{L} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = 0$$

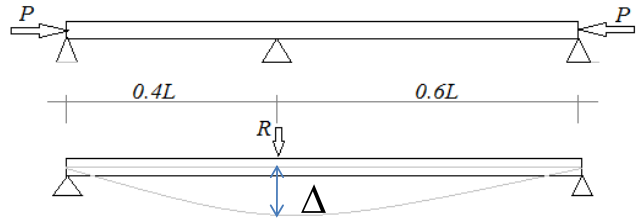
$$P_{cr1} = \frac{8.91EI}{L^2} \quad \text{Err: } \%5.5$$

$$w = a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3$$

$$\phi_1(x) = \sin \frac{\pi x}{L} \quad \phi_2(x) = \sin \frac{2\pi x}{L} \quad \phi_3(x) = \sin \frac{3\pi x}{L}$$

$$w' = \frac{\pi}{L} a_1 \cos \frac{\pi x}{L} + \frac{2\pi}{L} a_2 \cos \frac{2\pi x}{L} + \frac{3\pi}{L} a_3 \cos \frac{3\pi x}{L}$$

$$w'' = -\frac{\pi^2}{L^2} a_1 \sin \frac{\pi x}{L} - \frac{4\pi^2}{L^2} a_2 \sin \frac{2\pi x}{L} - \frac{9\pi^2}{L^2} a_3 \sin \frac{3\pi x}{L}$$



$$V = U + V_p + V_R = \int_0^L \frac{EI}{2} (\sum a_i \phi_i''(x))^2 dx - \frac{P}{2} \int_0^L (\sum a_i \phi_i'(x))^2 dx - R\Delta$$

$$V = \frac{EI}{2} \frac{\pi^4}{L^4} \frac{L}{2} (a_1^2 + 16a_2^2 + 81a_3^2) - \frac{P}{2} \frac{\pi^2}{L^2} \frac{L}{2} (a_1^2 + 4a_2^2 + 9a_3^2) - R\Delta$$

$$w(x=0.4L) = \Delta \quad \Delta = 0.9511a_1 + 0.5878a_2 - 0.5878a_3 = 0$$

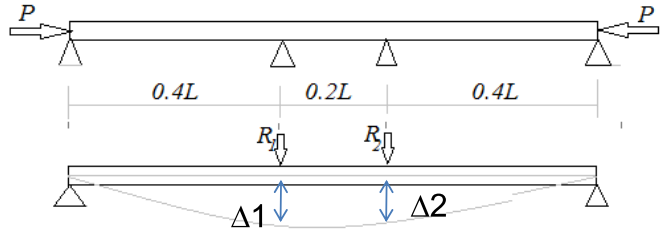
$$\frac{\partial V}{\partial a_i} = 0 \quad i=1,2,3$$

$$\begin{bmatrix} 1-\alpha & 0 & 0 & 0.9511 \\ & 4(4-\alpha) & 0 & 0.5878 \\ & & 9(9-\alpha) & -0.5878 \\ & & & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ r \end{Bmatrix} = 0 \quad \alpha = \frac{P}{P_c} \quad r = -\frac{2L^3}{\pi^4 EI} R$$

$$Det = 0 \quad \alpha^2 - 12.45\alpha + 32.542 = 0 \quad \alpha_1 = 3.733 \quad \& \quad \alpha_2 = 8.717$$

$$P_{cr1} = 3.733 \frac{\pi^2 EI}{L^2} \quad \text{Exact Solution} = 3.725 \frac{\pi^2 EI}{L^2}$$

$$w = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L}$$



$$V = U + V_p + V_R = \int_0^L \frac{EI}{2} (\sum a_i \phi_i''(x))^2 dx - \frac{P}{2} \int_0^L (\sum a_i \phi_i'(x))^2 dx - R_1 \Delta_1 - R_2 \Delta_2$$

$$\frac{\partial V}{\partial a_i} = 0 \quad i=1,2,3$$

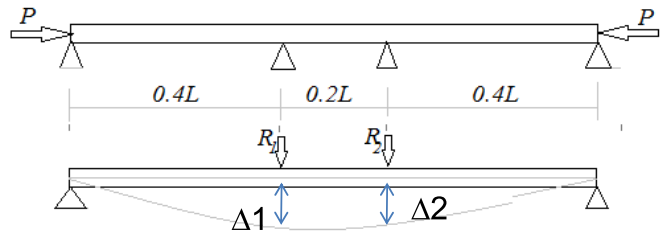
$$w(x=0.4L) = \Delta_1 \quad \Delta_1 = 0.9511a_1 + 0.5878a_2 - 0.5878a_3 = 0$$

$$w(x=0.6L) = \Delta_2 \quad \Delta_2 = 0.9511a_1 - 0.5878a_2 - 0.5878a_3 = 0$$

$$\begin{bmatrix} 1-\alpha & 0 & 0 & 0.9511 & 0.9511 \\ & 4(4-\alpha) & 0 & 0.5878 & -0.5878 \\ & & 9(9-\alpha) & -0.5878 & -0.5878 \\ 0.9511 & 0.5878 & -0.5878 & 0 & 0 \\ 0.9511 & -0.5878 & -0.5878 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ r_1 \\ r_2 \end{Bmatrix} = 0$$

$$\alpha = \frac{P}{P_c} \quad r_1 = -\frac{2L^3}{\pi^4 EI} R_1 \quad r_2 = -\frac{2L^3}{\pi^4 EI} R_2$$

$$Det = 0 \quad \alpha = 9 \quad P_{cr1} = 9 \frac{\pi^2 EI}{L^2}$$



$$w = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + a_3 \sin \frac{3\pi x}{L} + a_4 \sin \frac{4\pi x}{L}$$

$$V = U + V_P + V_R = \int_0^L \frac{EI}{2} \left(\sum a_i \phi_i''(x) \right)^2 dx - \frac{P}{2} \int_0^L \left(\sum a_i \phi_i'(x) \right)^2 dx - R_1 \Delta_1 - R_2 \Delta_2$$

$$\frac{\partial V}{\partial a_i} = 0 \quad i = 1, 2, 3, 4$$

$$w(x = 0.4L) = \Delta_1 \quad \Delta_1 = 0.9511a_1 + 0.5878a_2 - 0.5878a_3 - 0.9511a_4 = 0$$

$$w(x = 0.6L) = \Delta_2 \quad \Delta_2 = 0.9511a_1 - 0.5878a_2 - 0.5878a_3 + 0.9511a_4 = 0$$

$$P_{cr1} = 9 \frac{\pi^2 EI}{L^2} \quad \& \quad P_{cr1} = 13.041 \frac{\pi^2 EI}{L^2}$$

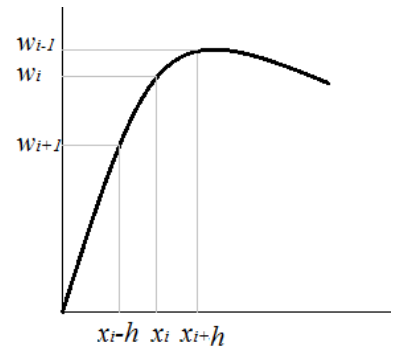
Finite difference method:

$$\frac{dw}{dx} = \lim_{h \rightarrow 0} \frac{w_{(x+h)} - w_{(x)}}{h} \approx \frac{w_{(x+h)} - w_{(x)}}{h} \Rightarrow$$

$$\left. \frac{dw}{dx} \right|_{x=x_i} \approx \frac{w_{i+1} - w_i}{h} = \Delta w_i \quad \text{foreward difference (F.D.)}$$

$$\left. \frac{dw}{dx} \right|_{x=x_i} \approx \frac{w_{i+1} - w_{i-1}}{2h} = \Delta w_i \quad \text{central difference (C.D.)}$$

$$\left. \frac{dw}{dx} \right|_{x=x_i} \approx \frac{w_i - w_{i-1}}{h} = \Delta w_i \quad \text{backward difference (B.D.)}$$

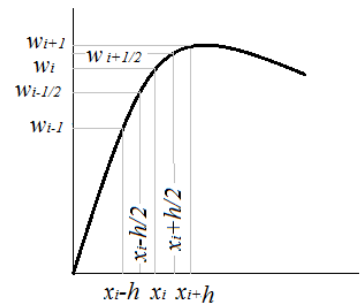


$$\left. \frac{d^2 w}{dx^2} \right|_{x=x_i} \approx \Delta^2 w_i = \Delta(\Delta w_i) = \Delta \frac{w_{i+1/2} - w_{i-1/2}}{h} = \frac{\frac{w_{i+1} - w_i}{h} - \frac{w_i - w_{i-1}}{h}}{h}$$

$$\Delta^2 w_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$$

$$\left. \frac{d^3 w}{dx^3} \right|_{x=x_i} \approx \Delta^3 w_i = \frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3}$$

$$\left. \frac{d^4 w}{dx^4} \right|_{x=x_i} \approx \Delta^4 w_i = \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4}$$



Or:
$$\begin{bmatrix} 2h\Delta w_i \\ h^2\Delta^2 w_i \\ 2h^3\Delta^3 w_i \\ h^4\Delta^4 w_i \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ -1 & 2 & 0 & -2 & 1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} w_{i-2} \\ w_{i-1} \\ w_i \\ w_{i+1} \\ w_{i+2} \end{bmatrix} \quad C. D.$$

$$\begin{bmatrix} h\Delta w_i \\ h^2\Delta^2 w_i \\ h^3\Delta^3 w_i \\ h^4\Delta^4 w_i \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} w_i \\ w_{i+1} \\ w_{i+2} \\ w_{i+3} \\ w_{i+4} \end{bmatrix} \quad F. D.$$

$$\begin{bmatrix} h\Delta w_i \\ h^2\Delta^2 w_i \\ h^3\Delta^3 w_i \\ h^4\Delta^4 w_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & -1 & 3 & -3 & 1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} w_{i-4} \\ w_{i-3} \\ w_{i-2} \\ w_{i-1} \\ w_i \end{bmatrix} \quad B. D.$$

Example:

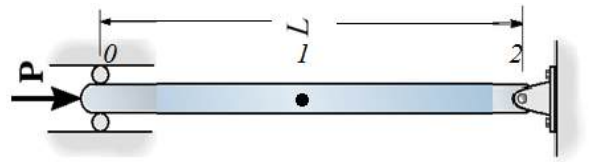
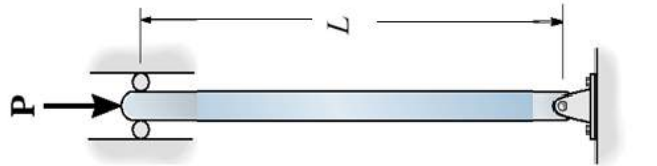
$$\frac{d^2 w}{dx^2} + \left(\frac{P}{EI} \right) w = 0$$

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \mu^2 w_i = 0 \quad \text{or:}$$

$$w_{i+1} + (\mu^2 h^2 - 2)w_i + w_{i-1} = 0$$

Point 1: $w_0 + (\mu^2 h^2 - 2)w_1 + w_2 = 0$

$$\mu^2 h^2 = 2 \Rightarrow P_{cr} = \frac{8EI}{L^2}$$



Symmetric Consideration:

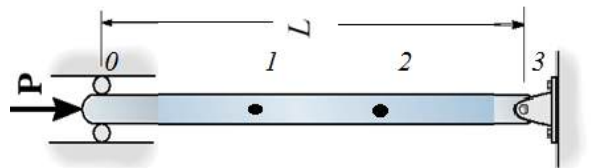
Point 1: $w_0 + (\mu^2 h^2 - 2)w_1 + w_2 = 0$

$$\mu^2 h^2 = 1 \Rightarrow P_{cr} = \frac{9EI}{L^2}$$

Without Symmetric Consideration:

Point 1: $w_0 + (\mu^2 h^2 - 2)w_1 + w_2 = 0$ Point 2: $w_1 + (\mu^2 h^2 - 2)w_2 + w_3 = 0$

$$(\mu^2 h^2 - 2)w_1 + w_2 = 0 \quad \& \quad w_1 + (\mu^2 h^2 - 2)w_2 = 0 \quad \mu^2 h^2 - 2 = \pm 1 \Rightarrow P_{cr1} = \frac{9EI}{L^2} \quad \& \quad P_{cr1} = \frac{27EI}{L^2}$$



Richardson Extrapolation:

$$Err = P_{exact} - P_{cr} \propto h^2 \quad \text{or:} \quad Err \approx ch^2$$

$$Err_1 \approx P_{approx} - P_{cr1} = c \left(\frac{L}{n_1} \right)^2 \quad \& \quad Err_2 \approx P_{approx} - P_{cr2} = c \left(\frac{L}{n_2} \right)^2 \Rightarrow P_{approx} = \frac{n_1^2 P_{cr1} - n_2^2 P_{cr2}}{n_1^2 - n_2^2}$$

	$P_{cr} L^2 / EI$	Err.
n=2	8	19
n=3	9	9
n=4	9.373	5
n=5	9.549	3
n1=2	9.8	0.7
n2=3		
n1=3	9.852	0.18
n2=4		
n1=4	9.861	0,08
n2=5		
n1=2	9.844	0.26
n2=5		

$$\Delta w_i = \frac{w_{i+1} - w_{i-1}}{(1+\alpha)h} \quad (C.D.)$$

$$\Delta^2 w_i = \frac{2(w_{i+1} - (1+\alpha)w_i + \alpha w_{i-1}))}{\alpha(1+\alpha)h^2}$$

Example:

$$\frac{d^2 w}{dx^2} + \left(\frac{P}{EI} \right) w = 0$$

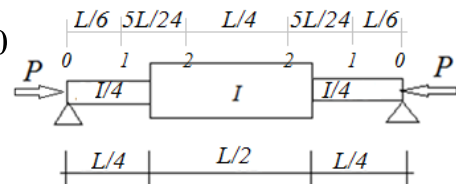
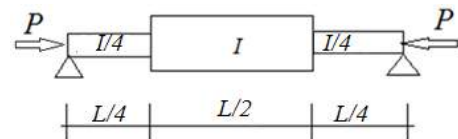
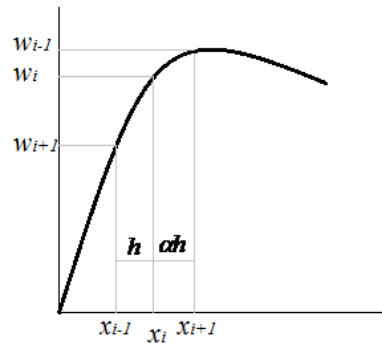
$$\frac{2(w_{i+1} - (1+\alpha_i)w_i + \alpha w_{i-1}))}{\alpha_i(1+\alpha_i)h_i^2} + \mu_i^2 w_i = 0 \quad \text{or:}$$

$$2w_{i+1} + (\alpha_i(1+\alpha_i)\mu_i^2 h_i^2 - 2(1+\alpha_i))w_i + 2\alpha_i w_{i-1} = 0$$

Point 1: $\alpha_1 = 1.25 \quad h_1 = L/6 \quad \mu_1 = \sqrt{4P/EI} = 2\mu$

$$2w_2 + (1.25(4)(1+1.25)\mu^2 L^2 / 36 - 2(1+1.25))w_1 = 0$$

$$2w_2 + (0.3125\mu^2 L^2 - 4.5)w_1 = 0$$



$$\text{Point } 2: \alpha_2 = 1.2 \quad h_2 = 5L/24 \quad \mu_2 = \mu$$

$$2w_2 + (1.2(25)(1+1.2)\mu^2 L^2 / 576 - 2(1+1.2))w_2 + 2.4w_1 = 0$$

$$(0.1130\mu^2 L^2 - 2.4)w_2 + 2.4w_1 = 0$$

$$(0.1130\mu^2 L^2 - 2.4)(0.3125\mu^2 L^2 - 4.5) - 4.8 = 0$$

$$0.0353\beta^2 - 1.2585\beta + 6 = 0 \quad \beta = 5.67 \quad P_{cr} = 0.574 \frac{\pi^2 EI}{L^2}$$

$$P_{exact} = 0.63 \frac{\pi^2 EI}{L^2} \quad Err: 12.2\%$$

for Constant t EI and P

$$w^{IV} + \left(\frac{P}{EI} \right) w'' = q$$

$$w_{i-2} + (\mu^2 h^2 - 4)w_{i-1} + (-2\mu^2 h^2 + 6)w_i + (\mu^2 h^2 - 4)w_{i+1} + w_{i+2} = q_i h^4 / EI$$

For variable EI and P

$$(EIw'')'' + (Pw')' = q$$

$$\Delta^2 (EIw'')_i = \frac{(EIw'')_{i-1} - 2(EIw'')_i + (EIw'')_{i+1}}{h^2}$$

$$(EIw'')_{i-1} \approx EI_{i-1} \frac{w_i - 2w_{i-1} + w_{i-2}}{h^2} \quad (EIw'')_i \approx EI_i \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \quad (EIw'')_{i+1} \approx EI_{i+1} \frac{w_{i+2} - 2w_{i+1} + w_i}{h^2}$$

$$\Delta^2 (EIw'')_i = \frac{EI_{i-1} w_{i-2} + (-2EI_{i-1} + 2EI_i)w_{i-1} + (EI_{i-1} - 4EI_i + EI_{i+1})w_i + (2EI_i + 2EI_{i+1})w_{i+1} + EI_{i+1} w_{i+2}}{h^4}$$

$$\Delta(Pw') = \frac{-(Pw')_{i-1/2} + (Pw')_{i+1/2}}{h}$$

$$(Pw')_{i-1/2} = \frac{P_{i-1/2}}{h} (-w_{i-1} + w_i) \quad (Pw')_{i+1/2} = \frac{P_{i+1/2}}{h} (-w_i + w_{i+1})$$

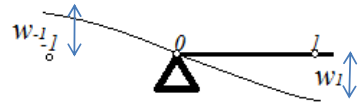
$$\Delta(Pw')_i = \frac{P_{i-1/2} w_{i-1} - (P_{i-1/2} + P_{i+1/2})w_i + P_{i+1/2} w_{i+1}}{h^2}$$

$\frac{EI_{i-1} w_{i-2} + (-2EI_{i-1} - 2EI_i + h^2 P_{i-1/2})w_{i-1} + (EI_{i-1} + 4EI_i + EI_{i+1} - h^2 (P_{i-1/2} + P_{i+1/2}))w_i}{h^4}$ $+ \frac{(-2EI_i - 2EI_{i+1} + h^2 P_{i+1/2})w_{i+1} + EI_{i+1} w_{i+2}}{h^4} = q_i$
--

Boundary conditions:

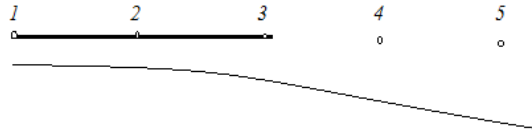
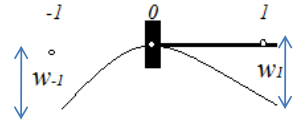
pin end :

$$\Delta^2 w)_0 = 0 \quad \frac{w_{-1} - 2w_0 + w_1}{h} = 0 \xrightarrow{w_0=0} w_{-1} = -w_1$$



fix end :

$$\Delta w)_0 = 0 \quad \frac{-w_{-1} + w_1}{2h} = 0 \longrightarrow w_{-1} = w_1$$



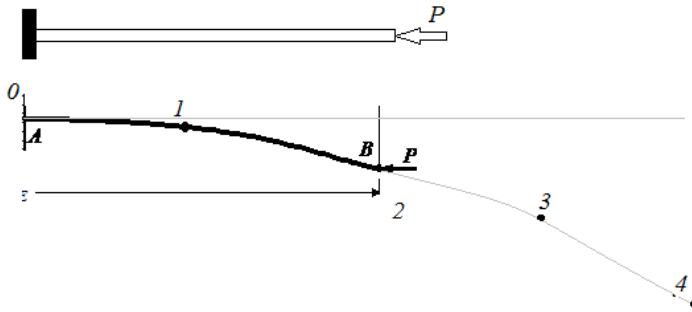
free end :

$$V)_3 = EI \frac{d^3 w}{dx^3} + P \frac{dw}{dx} = 0 \quad \Rightarrow \Delta^3 w + \mu^2 \Delta w = 0 \quad w_5 - 2w_4 + 2w_2 - w_1 + \mu^2 h^2 (-w_2 + w_4) = 0$$

$$M)_3 = -EI \frac{d^2 w}{dx^2} = -EI \Delta^2 w = \frac{w_2 - 2w_3 + w_4}{h} = 0 \quad w_4 = -w_2 + 2w_3$$

$$w_5 = (\mu^2 h^2 - 2)(2w_3) + 2(2 - \mu^2 h^2)w_2 - w_1$$

بار بحرانی تیر شکل را به روش تفاوتهای محدود بدست آورید



$$w_{i-2} + (\mu^2 h^2 - 4)w_{i-1} + (-2\mu^2 h^2 + 6)w_i + (\mu^2 h^2 - 4)w_{i+1} + w_{i+2} = q_i h^4$$

$$\text{Point 1: } w_{-1} + (\mu^2 h^2 - 4)w_0 + (-2\mu^2 h^2 + 6)w_1 + (\mu^2 h^2 - 4)w_2 + w_3 = 0$$

$$(-2\mu^2 h^2 + 6)w_1 + (\mu^2 h^2 - 2)w_2 = 0$$

$$\text{Point 2: } w_0 + (\mu^2 h^2 - 4)w_1 + (-2\mu^2 h^2 + 6)w_2 + (\mu^2 h^2 - 4)w_3 + w_4 = 0$$

$$2(\mu^2 h^2 - 2)w_1 + (-2\mu^2 h^2 + 6)w_2 = 0$$

$$(6 - 2\alpha)^2 - 2(\alpha - 2)^2 = 0$$

$$w_3 = -w_1 + 2w_2 \quad \alpha^2 - 8\alpha + 14 = 0 \quad \alpha = 4 \pm \sqrt{2}$$

$$w_4 = 2(\mu^2 h^2 - 2)(w_2 - w_1)$$

Stiffness matrix of a strut: Rayleigh-Ritz method

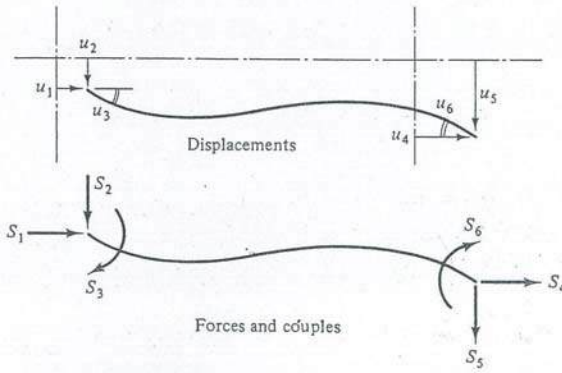


Fig. 1.22 Notation for displacements (u) and forces or couples (S).

$$w = ax^3 + bx^2 + cx + d.$$

$$x = 0, \quad w = u_2 = d,$$

$$x = 0, \quad w' = u_3 = c,$$

$$x = L, \quad w = u_5 = aL^3 + bL^2 + cL + d,$$

$$x = L, \quad w' = u_6 = 3aL^2 + 2bL + c.$$

$$w = (2u_2 - 2u_5 + Lu_3 + Lu_6)\left(\frac{x}{L}\right)^3 + (-3u_2 + 3u_5 - 2Lu_3 - Lu_6)\left(\frac{x}{L}\right)^2 + u_3x + u_2$$

$$\text{or } w = u_2\phi_2(x) + u_3\phi_3(x) + u_5\phi_5(x) + u_6\phi_6(x)$$

$$\begin{aligned} \phi_2 &= 2\xi^3 - 3\xi^2 + 1, & \phi_3 &= L(\xi^3 - 2\xi^2 + \xi), & \xi &= x/L. \\ \phi_5 &= -2\xi^3 + 3\xi^2, & \phi_6 &= L(\xi^3 - \xi^2), \end{aligned}$$

$$\begin{aligned} \phi_2' &= 6(\xi^2 - \xi)/L, & \phi_3' &= L(3\xi^2 - 4\xi + 1)/L, \\ \phi_5' &= 6(-\xi^2 + \xi)/L, & \phi_6' &= L(3\xi^2 - 2\xi)/L. \end{aligned}$$

$$\begin{aligned} v_{22} &= \int_0^L (\phi_2')^2 dx = \frac{36}{L^2} \int_0^L (\xi^2 - \xi)^2 dx \\ &= \frac{36}{L} \int_0^1 (\xi^4 - 2\xi^3 + \xi^2) d\xi = \frac{6}{5} \frac{1}{L}, \end{aligned}$$

$$\begin{aligned} v_{23} &= \int_0^L \phi_2' \phi_3' dx = \frac{6}{L} \int_0^L (\xi^2 - \xi)(3\xi^2 - 4\xi + 1) dx \\ &= 6 \int_0^1 (3\xi^4 - 7\xi^3 + 5\xi^2 - \xi) d\xi = \frac{1}{10}. \end{aligned}$$

$$\begin{aligned} v_{26} &= \frac{1}{10} = -v_{56} = -v_{35}; & v_{33} &= \frac{2}{15} L = v_{66}; \\ v_{36} &= -\frac{L}{30}; & v_{55} &= +\frac{6}{5} \frac{1}{L} = -v_{25}. \end{aligned}$$

The second derivatives of the displacement function are

$$\begin{aligned} \phi_2'' &= 6(2\xi - 1)/L^2, & \phi_3'' &= 2(3\xi - 2)/L, \\ \phi_5'' &= 6(-2\xi + 1)/L^2, & \phi_6'' &= 2(3\xi - 1)/L. \end{aligned}$$

$$\begin{aligned} u_{22} &= \int_0^L EI(\phi_2'')^2 dx = \frac{36EI}{L^4} \int_0^L (2\xi - 1)^2 dx \\ &= \frac{36EI}{L^3} \int_0^1 (4\xi^2 - 4\xi + 1) d\xi = \frac{12EI}{L^3}, \end{aligned}$$

$$\begin{aligned} u_{23} &= \int_0^L EI\phi_2''\phi_3'' dx = \frac{6EI}{L^3} \int_0^L (2\xi - 1)(3\xi - 2) dx \\ &= \frac{12EI}{L^2} \int_0^1 (6\xi^2 - 7\xi + 2) d\xi = \frac{6EI}{L^2}. \end{aligned}$$

$$\begin{aligned} u_{26} &= \frac{6EI}{L^2} = -u_{35} = -u_{56}; & u_{33} &= \frac{4EI}{L} = u_{66}; \\ u_{36} &= \frac{2EI}{L}; & u_{55} &= \frac{12EI}{L^3} = -u_{25}. \end{aligned}$$

$$u_{26} = \frac{6EI}{L^2} = -u_{35} = -u_{56}; \quad u_{33} = \frac{4EI}{L} = u_{66};$$

$$u_{36} = \frac{2EI}{L}; \quad u_{55} = \frac{12EI}{L^3} = -u_{25}$$

The forces and couples S_2, S_3, S_5, S_6 are 'transverse forces' in the sense that they cause the strut to deviate from the straight condition. The potential energy associated with them is $-(S_2 u_2 + S_3 u_3 + S_5 u_5 + S_6 u_6)$ and the corresponding generalized forces, according to the definition used in setting up Eq. (1.111) and (1.113), are:†

$$q_2 = S_2; \quad q_3 = S_3; \quad q_5 = S_5; \quad q_6 = S_6; \quad (1.133)$$

The results (1.130), (1.132), and (1.133) may now be gathered together to construct an equation comparable with Eq. (1.117):

$$(\mathbf{k}_E + \mathbf{k}_G)\mathbf{u} = \mathbf{S} \quad (1.134)$$

$$\mathbf{k}_E = \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & -\frac{12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & -\frac{6}{L} & 2 \\ -\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2} & -\frac{6}{L} \\ \frac{6}{L} & 2 & -\frac{6}{L} & 4 \end{bmatrix}, \quad \mathbf{k}_E = \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & -\frac{12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & -\frac{6}{L} & 2 \\ -\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2} & -\frac{6}{L} \\ \frac{6}{L} & 2 & -\frac{6}{L} & 4 \end{bmatrix},$$

$$\text{and} \quad \mathbf{u} = \{u_2 \ u_3 \ u_5 \ u_6\}; \quad \mathbf{S} = \{S_2 \ S_3 \ S_5 \ S_6\}. \quad (1.135b)$$

The relationships between the axial forces S_1, S_4 and the axial displacements, u_1, u_4 (Fig. 1.22) are determined solely by the axial stiffness of the strut, EA :

$$S_1 = \frac{EA}{L} (u_1 - u_4) = -S_4. \quad (1.136)$$

Equations (1.135) and (1.136) may be combined to give a single result which is of the same form as Eq. (1.134) except that the matrices are now as follows:

$$k_E = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

$$k_G = -\frac{P}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix}$$

$$\mathbf{u} = \{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}, \quad \mathbf{S} = \{S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6\}.$$

Slope-Deflection Equation

(with Axial Force)

$$M_{ab} = \frac{EI}{\ell} \left[S_1 \theta_a + S_2 \theta_b - (S_1 + S_2) \frac{\Delta}{\ell} \right]$$

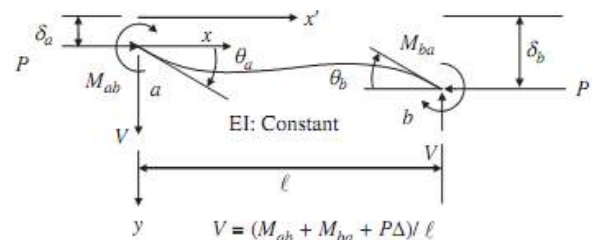
$$M_{ba} = \frac{EI}{\ell} \left[S_2 \theta_a + S_1 \theta_b - (S_1 + S_2) \frac{\Delta}{\ell} \right]$$

$$M_{ab} = S \frac{EI}{\ell} \left[\theta_a + C \theta_b - (1+C) \frac{\Delta}{\ell} \right]$$

$$M_{ba} = S \frac{EI}{\ell} \left[C \theta_a + \theta_b - (1+C) \frac{\Delta}{\ell} \right]$$

$$S'_1 = S' = \frac{\beta(\beta \cosh \beta - \sinh \beta)}{2 - 2 \cosh \beta + \beta \sinh \beta}$$

$$\frac{\beta(\sinh \beta - \beta)}{2 - 2 \cosh \beta + \beta \sinh \beta}$$



$$S_1 = S = \frac{\beta(\beta \cos \beta - \sin \beta)}{2 \cos \beta + \beta \sin \beta - 2}$$

$$S_2 = SC = \frac{(\sin \beta - \beta)}{2 \cos \beta + \beta \sin \beta - 2}$$

$$\beta = \mu L$$

Stability functions:

$$q = \frac{PL^2}{\pi^2 EI}$$

Tensile Axial Force, ($q < 0$):

$$s = \frac{\psi^2 \cosh \psi - \psi \sinh \psi}{2 - 2 \cosh \psi - \psi \sinh \psi}$$

$$SC = \frac{\psi \sinh \psi - \psi^2}{2 - 2 \cosh \psi - \psi \sinh \psi}$$

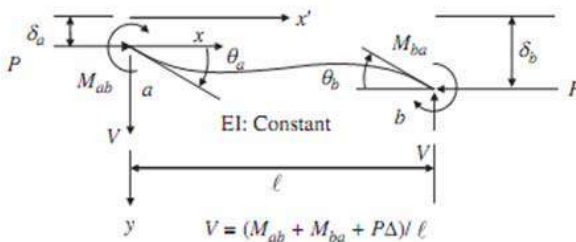
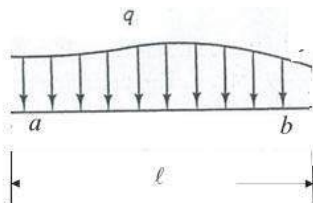
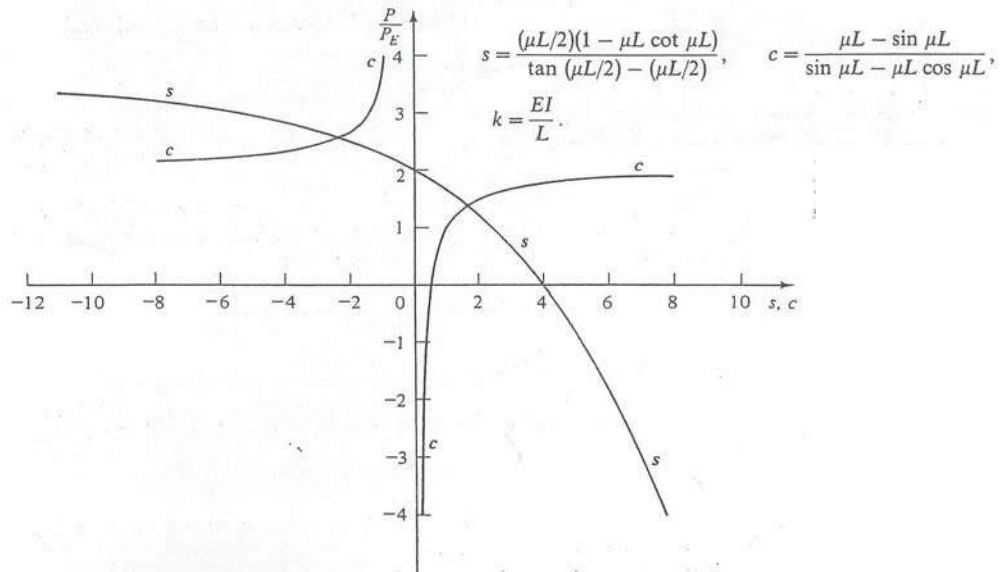
in which $\psi^2 = -\pi^2 q \dots\dots$

Compressive Axial Force, ($q > 0$):

$$S = \frac{\phi \sin \phi - \phi^2 \cos \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$SC = \frac{\phi^2 - \phi \sin \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

in which $\phi^2 = \pi^2 q \dots\dots$



$$M'_{ab} = Sk\theta_a \quad \& \quad M'_{ba} = SkC\theta_a$$

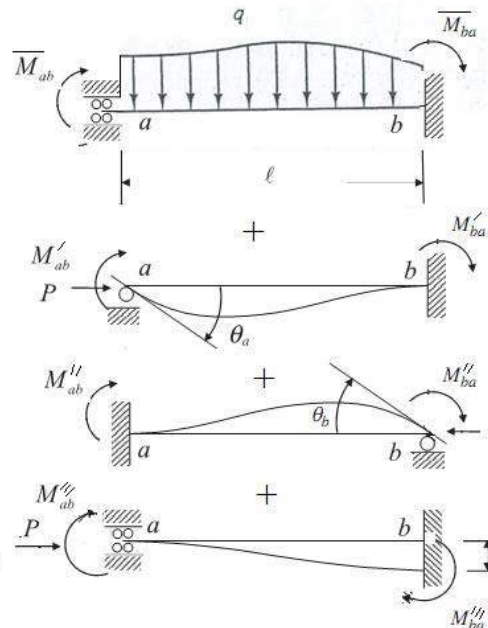
$$M''_{ab} = SkC\theta_b \quad \& \quad M''_{ba} = Sk\theta_b$$

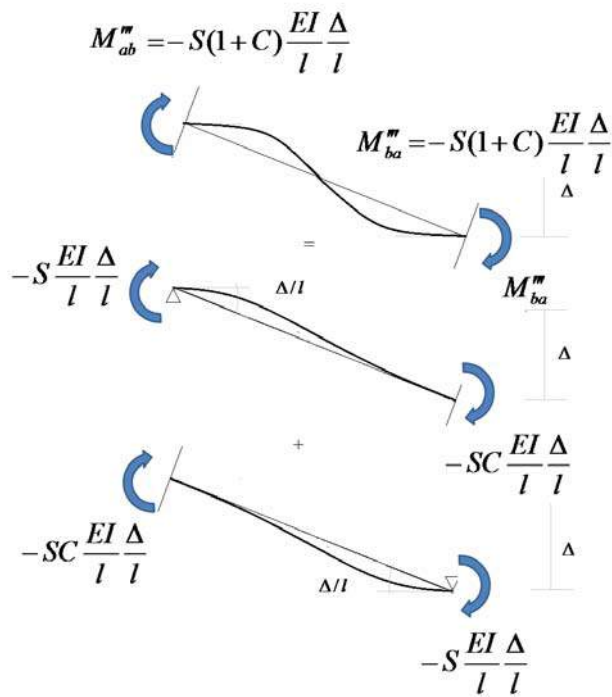
$$s = \frac{(\mu L / 2)(1 - \mu L \cot \mu L)}{\tan (\mu L / 2) - (\mu L / 2)}, \quad c = \frac{\mu L - \sin \mu L}{\sin \mu L - \mu L \cos \mu L},$$

$$k = \frac{EI}{L}.$$

Slope-Deflection Equation

(with Axial Force)





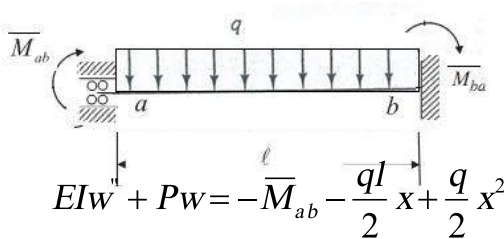
$$M_{ab} = \bar{M}_{ab} + M'_{ab} + M''_{ab} + M'''_{ab}$$

$$M_{ba} = \bar{M}_{ba} + M'_{ba} + M''_{ba} + M'''_{ba}$$

$$\begin{aligned} M_{ab} &= \bar{M}_{ab} + Sk(\theta_a + C\theta_b - (1+C)\frac{\Delta}{l}) \\ M_{ba} &= \bar{M}_{ba} + Sk(C\theta_a + \theta_b - (1+C)\frac{\Delta}{l}) \end{aligned}$$

Fix End Moment

(uniform distributed load)



$$EIw'' + Pw = -\bar{M}_{ab} - \frac{ql}{2}x + \frac{q}{2}x^2$$

$$w = A \cos \mu x + B \sin \mu x - \frac{\bar{M}_{ab}}{P} + \frac{q}{2P}(x^2 - lx - \frac{2}{\mu^2})$$

$$@ x=0 \quad w=0, @ x=l \quad w=0 \quad \& @ x=0 \quad w'=0 \quad \& @ x=l \quad w'=0$$

$$A = \frac{ql}{2P\mu} \cot \frac{\mu l}{2}, B = \frac{ql}{2P\mu} \quad \bar{M}_{ab} = -\bar{M}_{ba} = -\frac{ql^2}{12} \beta_5 \quad M|_{x=l/2} = \frac{ql^2}{24} \beta_6$$

$$\beta_5 = \frac{3}{(\mu l/2)^2} (1 - \frac{\mu l}{2} \cot \frac{\mu l}{2}) \quad \beta_6 = \frac{3}{(\mu l/2)^2} (\frac{\mu l}{2} \cot \frac{\mu l}{2} - 1)$$

Table 2.15 Fixed-end moments for struts with various types of transverse load (from Eq. (2.250)). Positive end-moments cause tension on the underside of the beam-column. The functions d, e, f, g are given by Eqs. (2.249) and (2.251). $\mu = \sqrt{P/EI}$; $\mu L = \pi\sqrt{P/P_E}$.

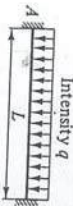
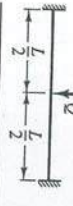
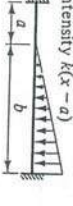
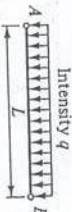
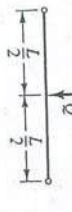
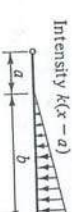
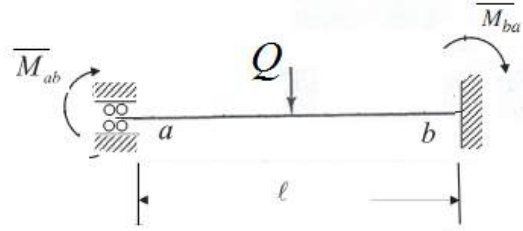
Type of load	Fixed-end moments
	$M_a = M_b = -\frac{q}{\mu^2} \left(1 - \frac{\mu L}{2} \cot \frac{\mu L}{2} \right)$
	$M_a = M_b = -\frac{Q}{2\mu} \tan \frac{\mu L}{4}$
	$M_a = \frac{C}{2g} \{ e \sin \mu b - f(1 - \cos \mu b) \}$ $M_b = \frac{C}{2g} \{ -d \sin \mu b + f(1 + \cos \mu b) - \mu L \cos \mu b \}$
	$M_a = \frac{Q}{2\mu g} \{ f(\mu b - \sin \mu b) - e(1 - \cos \mu b) \}$ $M_b = \frac{Q}{2\mu g} \{ -f(\mu b + \sin \mu b) + d(1 - \cos \mu b) + \mu L \sin \mu b \}$
	$M_a = \frac{q}{2\mu^2 g} \left\{ -e(\mu b - \sin \mu b) - f(1 - \cos \mu b - \frac{\mu^2 b^2}{2}) \right\}$ $M_b = \frac{q}{2\mu^2 g} \left\{ d(\mu b - \sin \mu b) - f(1 - \cos \mu b + \frac{\mu^2 b^2}{2}) + \mu L(1 - \cos \mu b) \right\}$
	$M_a = \frac{k}{2\mu^3 g} \left\{ \int (\sin \mu b + \frac{\mu^3 b^3}{6} - \mu b) + e \left(1 - \cos \mu b - \frac{\mu^2 b^2}{2} \right) \right\}$ $M_b = \frac{k}{2\mu^3 g} \left\{ \int (\sin \mu b - \frac{\mu^3 b^3}{6} - \mu b) - d \left(1 - \cos \mu b - \frac{\mu^2 b^2}{2} \right) + \mu^2 bL - \mu L \sin \mu b \right\}$
	$M_a = -\frac{PL\Delta f}{2Lg}$; $M_b = +\frac{PL\Delta f}{2Lg}$
	$M_a = +\frac{P}{2\mu g} (d\theta_a + e\theta_b)$ $M_b = -\frac{P}{2\mu g} (d\theta_b + e\theta_a)$

Table 2.16 End-rotations of pin-ended struts with various types of transverse load (from Eq. (2.248)). The functions d, e, f, g are given by Eqs. (2.249) and (2.251). $\mu = \sqrt{P/EI}$; $\mu L = \pi\sqrt{P/P_E}$.

Type of load	End-rotations (Positive clockwise)
	$\theta_a = -\theta_b = \frac{q}{P\mu} \left(\sec \frac{\mu L}{2} - \frac{\mu L}{2} \right)$
	$\theta_a = -\theta_b = \frac{Q}{2P} \left(\sec \frac{\mu L}{2} - 1 \right)$
	$\theta_a = \frac{C}{PL} \{ 1 - (1 + e) \cos \mu b \}$ $\theta_b = \frac{C}{PL} \{ 1 - (1 - d) \cos \mu b - \mu L \sin \mu b \}$
	$\theta_a = \frac{Q}{P\mu L} \{ -\mu b + (1 + e) \sin \mu b \}$ $\theta_b = \frac{Q}{P\mu L} \{ -\mu b + (1 - d) \sin \mu b + \mu L(1 - \cos \mu b) \}$
	$\theta_a = \frac{q}{P\mu^2 L} \left\{ -\frac{\mu^2 b^2}{2} + (1 + e)(1 - \cos \mu b) \right\}$ $\theta_b = \frac{q}{P\mu^2 L} \left\{ -\frac{\mu^2 b^2}{2} + (1 - d)(1 - \cos \mu b) + \mu L(\mu b - \sin \mu b) \right\}$
	$\theta_a = \frac{k}{P\mu^3 L} \left\{ -\frac{\mu^3 b^3}{6} + (1 + e)(\mu b - \sin \mu b) \right\}$ $\theta_b = \frac{k}{P\mu^3 L} \left\{ -\frac{\mu^3 b^3}{6} + (1 - d)(\mu b - \sin \mu b) + \mu L \left(\frac{\mu^2 b^2}{2} + \cos \mu b - 1 \right) \right\}$
	$\theta_a = \frac{1}{PL} (dM_a + eM_b)$ $\theta_b = -\frac{1}{PL} (eM_a + dM_b)$

Fix End Moment (central point load)

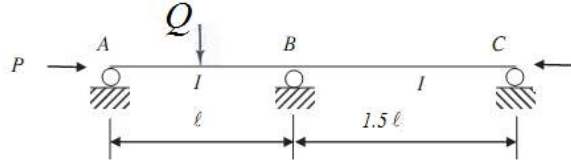
$$\bar{M}_{ab} = -\bar{M}_{ba} = -\frac{Q}{2\mu} \tan \frac{\mu l}{4}$$



Example:

$$P = 0.2 \frac{\pi^2 EI}{l^2}$$

$$\bar{M}_{ab} = -\bar{M}_{ba} = -\frac{Q}{2\mu} \tan \frac{\mu l}{4}$$



$$\bar{M}_{ab} = -\bar{M}_{ba} = 0.1304Ql$$

$$\mu_{ab} l_{ab} = \pi \sqrt{0.2} \Rightarrow s_{ab} = 3.7297 \quad c_{ab} = 0.555$$

$$\mu_{bc} l_{bc} = \pi \sqrt{0.45} \Rightarrow s_{ab} = 3.3697 \quad c_{ab} = 0.644$$

$$M_{AB} = \bar{M}_{AB} + S_{AB} k_{AB} (\theta_A + C_{AB} \theta_B - (1 + C_{AB}) \frac{\Delta_{AB}}{l_{AB}}) = \bar{M}_{AB} + S_{AB} k_{AB} (\theta_A + C_{AB} \theta_B)$$

$$M_{BA} = \bar{M}_{BA} + S_{AB} k_{AB} (C_{AB} \theta_A + \theta_B - (1 + C_{AB}) \frac{\Delta_{AB}}{l_{AB}}) = \bar{M}_{BA} + S_{AB} k_{AB} (C_{AB} \theta_A + \theta_B)$$

$$M_{BC} = S_{BC} k_{BC} (\theta_B + C_{BC} \theta_C)$$

$$M_{CB} = S_{BC} k_{BC} (C_{BC} \theta_B + \theta_C)$$

$$M_{AB} = -0.1304Ql + 3.7297k(\theta_A + 0.555\theta_B)$$

$$M_{BA} = 0.1304Ql + 3.7297k(0.555\theta_A + \theta_B)$$

$$M_{BC} = 3.3697 \frac{k}{1.5} (\theta_B + 0.644\theta_C)$$

$$M_{CB} = 3.3697 \frac{k}{1.5} (0.644\theta_B + \theta_C)$$

$$M_{AB} = -0.1304Ql + 3.7297k(\theta_A + 0.555\theta_B) = 0$$

$$M_{BC} + M_{BA} = 0.1304Ql + 3.7297k(0.555\theta_A + \theta_B) + 3.3697 \frac{k}{1.5} (\theta_B + 0.644\theta_C) = 0$$

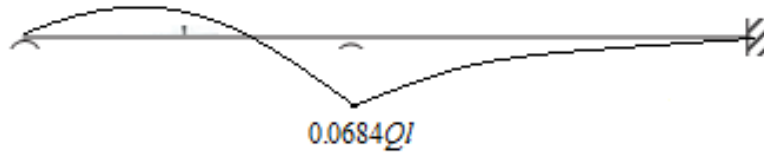
$$M_{CB} = 3.3697 \frac{k}{1.5} (0.644\theta_B + \theta_C) = 0 \quad \beta = Ql/k$$

$$\begin{bmatrix} 3.7297 & 2.070 & 0 \\ 2.070 & 5.976 & 1.4467 \\ 0 & 1.4467 & 2.2465 \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0.1304\beta \\ -0.1304\beta \\ 0 \end{bmatrix}$$

$$\theta_A = 0.0639 \frac{Ql^2}{EI} \quad \theta_B = -0.0521 \frac{Ql^2}{EI} \quad \theta_C = 0.0335 \frac{Ql^2}{EI}$$

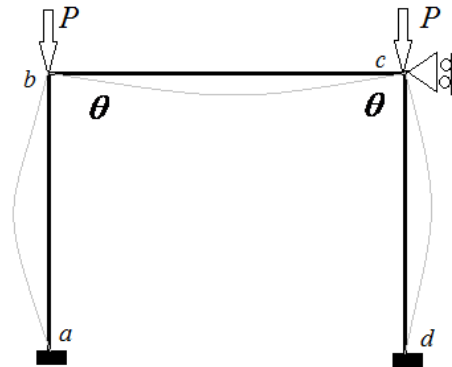
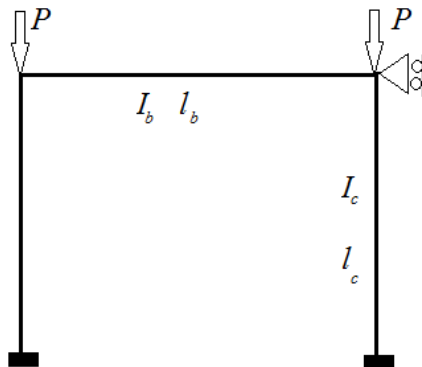
$$M_{BA} = 0.0684 Ql$$

$$M_{BC} = -0.0684 Ql$$



Moment Diagram

بار بحرانی قاب شکل را بدست آورید



$$M_{ba} = sk_c \theta$$

$$M_{bc} = 2k_b \theta$$

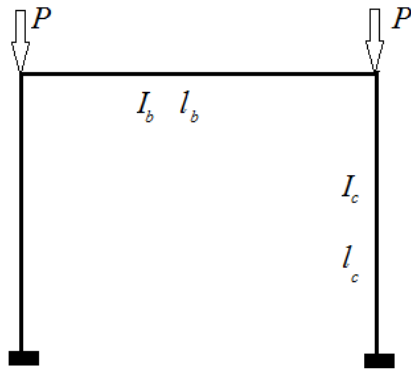
$$M_{ba} + M_{bc} = (sk_c + 2k_b) \theta = 0$$

$$s = -2 \frac{k_b}{k_c}$$

$$\text{for } I_b = I_c = I, \quad l_b = l_c = l, \quad s = -2$$

$$P_{cr} = 2.56 \frac{\pi^2 EI}{l^2}$$

بار بحرانی قاب شکل را بدست آورید



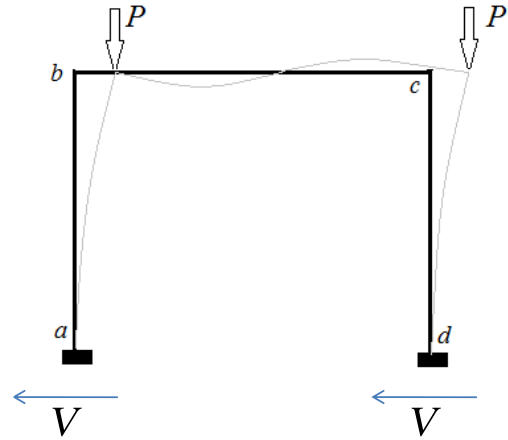
$$M_{ba} = sk_c \left(\theta - (1+c) \frac{\Delta}{l_c} \right)$$

$$M_{bc} = 6k_b \theta$$

$$M_{ba} + M_{bc} = sk_c \left(\theta - (1+c) \frac{\Delta}{l_c} \right) + 6k_b \theta = 0$$

$$\left(s + 6 \frac{k_b}{k_c} \right) \theta - s(1+c) \frac{\Delta}{l_c} = 0$$

$$s(1+c)\theta - (2s(1+c) - \mu^2 l_c^2) \frac{\Delta}{l_c} = 0$$

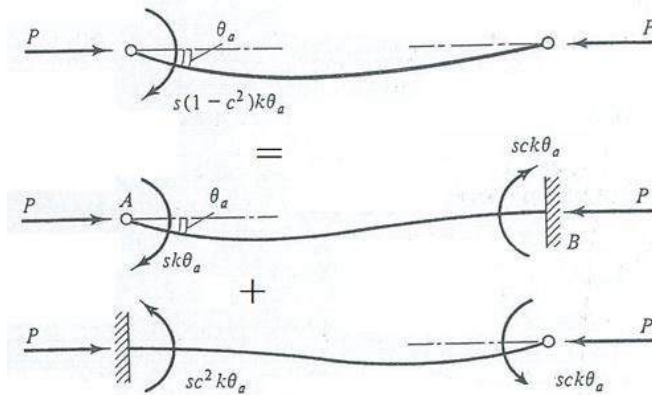


$$-Vl_c = M_{ab} + M_{ba} + P\Delta = 0$$

$$sk_c(1+c) \left(\theta - 2 \frac{\Delta}{l_c} \right) + P\Delta = 0$$

Moment distribution method

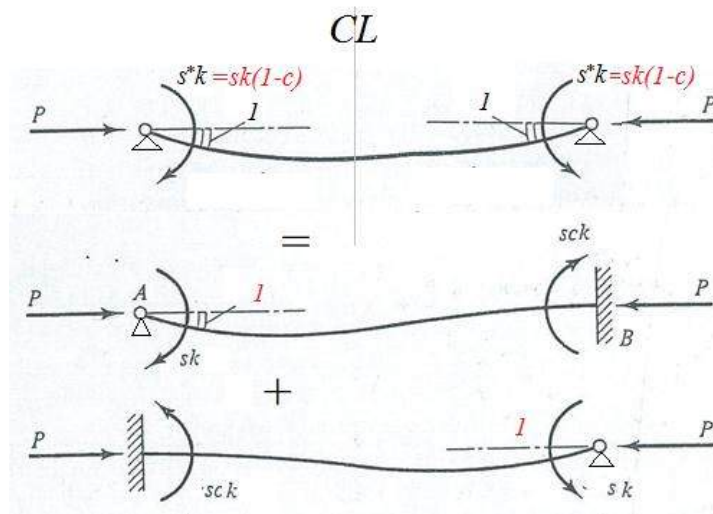
the stiffness of a strut when the far end is pinned.



$$S^* = s(1 - c^2)$$

Moment distribution method

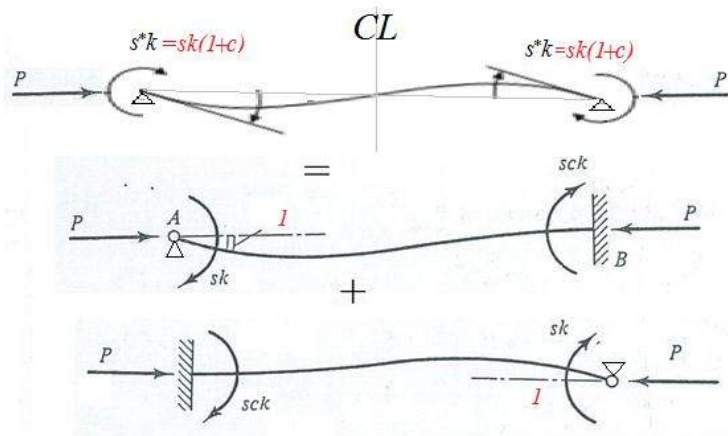
اصلاح سختی عضوی که توسط محور تقارن در سازه متقارن با بارگذاری متقارن نصف شده است



$$s^* = s(1 - c)$$

Moment distribution method

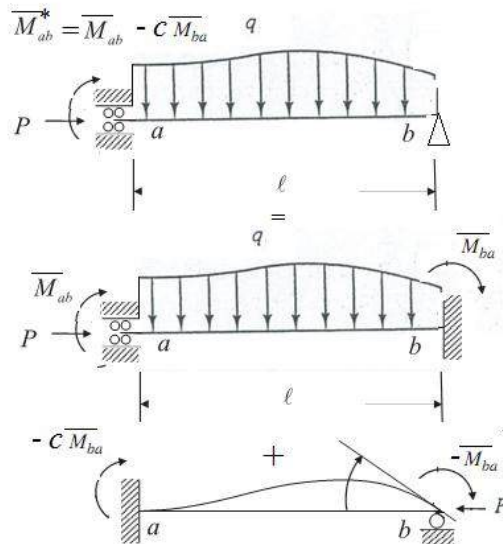
اصلاح سختی عضوی که توسط محور تقارن در سازه متقارن با بارگذاری معکوس نصف شده است



$$s^* = s(1 + c)$$

Moment distribution method

اصلاح ممان گیرداری عضو که یک انتهای آن مفصل است



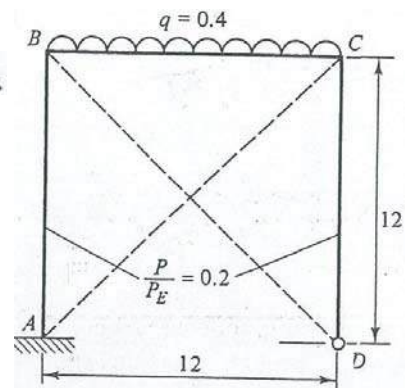
Example 1: $\frac{P}{P_E} = 0.2$, $s = 3.7297$, $c = 0.5550$.

$$\bar{M}_{ab} = -\bar{M}_{ba} = -\frac{ql^2}{12} \quad (sk)_{ba} = 3.7297k$$

$$\bar{M}_{cb} = -\bar{M}_{bc} = -4.8 \quad (s^*k)_{cd} = (s(1 - c^2)k)_{cd} = 2.5809k$$

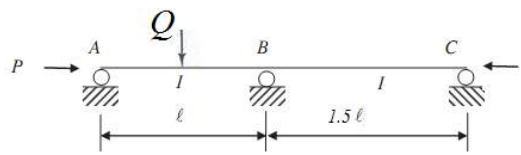
$$D_{ba} = \frac{3.7297}{4 + 3.7297} = 0.483 \quad D_{bc} = \frac{4}{4 + 3.7297} = 0.517$$

$$D_{cd} = \frac{2.5809}{4 + 2.5809} = 0.392 \quad D_{cb} = \frac{4}{4 + 2.5809} = 0.608$$



A	B		C		D	Distribution and carry-over factors
0.555	0.483 : 0.517		0.5	0.608 : 0.392		0
0	0	-4.80	+4.80	0	0	Fixed-end moments
		-1.46 ←	-2.92	-1.88	→ 0	
+1.68 ←	+3.02	+3.24 →	+1.62			
		-0.49 ←	-0.98	-0.64	→ 0	
+0.13 ←	+0.24	+0.25 →	+0.13			
		-0.04 ←	-0.08	-0.05	→ 0	
+0.01 ←	+0.02	+0.02 →	0			
+1.82	+3.28	-3.28	+2.57	-2.57	0	Result

Example1:



$$\mu_{ab} I_{ab} = \pi \sqrt{0.2} \Rightarrow s_{ab} = 3.7297 \quad c_{ab} = 0.555$$

$$s_{ab}^* = 3.7297(1 - (0.555)^2) = 2.5808$$

$$\mu_{bc} I_{bc} = \pi \sqrt{0.45} \Rightarrow s_{bc} = 3.3697 \quad c_{bc} = 0.644$$

$$s_{bc}^* = 3.3697(1 - (0.644)^2) = 1.9722$$

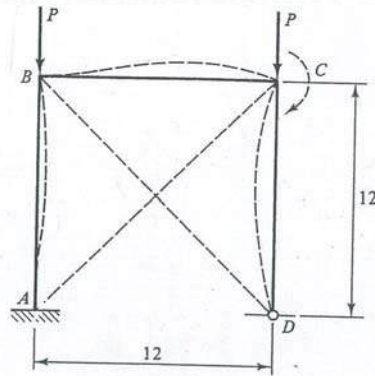
$$D_{ba} = \frac{2.5808}{2.5808 + 1.9722/1.5} = 0.662 \quad D_{bc} = 0.338$$

$$\bar{M}_{ab} = -\bar{M}_{ba} = -0.1304Ql$$

$$\bar{M}_{ba}^* = 0.1304Ql(1 + 0.555) = 0.2027Ql$$

0.662	0.338
0.2027Ql	
-0.1342Ql	-0.0685Ql

THE ELASTIC CRITICAL LOAD OF A FRAMEWORK (WITHOUT SWAY)



1. Assume a load that is associated with a certain value of P/P_E in the columns (say $P/P_E = 0.8$).
2. Apply an arbitrary couple at a joint such as C (say +100 units).
3. Carry out a complete distribution of moments in order to determine the out-of-balance moment which remains at C when the other joints have been balanced.
4. If this 'residual moment' is smaller than the original couple, the distribution is converging and the frame is stable; and vice versa.
5. Repeat the process at various loads in order to establish the critical value at which convergence fails.

$$P/P_E = 0.8 \quad s = 2.8159, \quad c = 0.8330.$$

$$\text{Stiffness ratios: } BA : BC = s : 4 = 2.8159 : 4 = 0.4131 : 0.5869, \\ \dagger CB : CD = 4 : s = 0.5869 : 0.4131.$$

A	B		C		D
0.8330	0.4131 : 0.5869		0.5	0.5869 : 0.4131	
			+100		
		-29.35 ←	-58.69	-41.31 →	-34.41 ← Unbalanced moment.
+10.10 ←	+12.12	+17.22 →	+8.61	+28.66 ←	+34.41

The residual moment at C is $+8.61 + 28.66 = +37.27$, which is numerically less than 100; the process therefore converges.

$$P/P_E = 1.5; \quad s = 1.4570, \quad c = 1.9731.$$

$$\text{Stiffness ratios: } BA : BC = s : 4 = 1.4570 : 4 = 0.2670 : 0.7330 \\ \dagger CB : CD = 4 : s = 0.7330 : 0.2670$$

A	B		C		D
1.9731	0.267 : 0.733	0.5	0.733 : 0.267	1.9731	
			+100		
		-36.65 ←	-73.3	-26.7 →	-52.68 ← Unbalanced moment.
+19.31 ←	+9.79	+26.86 →	+13.43	+103.95 ←	+52.68

The residual moment at C is $+13.43 + 103.95 = +117.38$, which is numerically greater than 100; the process therefore diverges.

MODIFICATIONS TO THE STIFFNESS AND CARRY-OVER FACTORS TO ALLOW FOR FLEXIBLE CONNECTIONS AND FOR RIGID ARMS AT THE ENDS OF MEMBERS

In Fig. 3.36 the member AB is attached to the adjacent structure by connections which are not completely rigid. Connections of this sort include those made by ordinary bolts and by rivets; even welded joints may not be rigid if insufficient stiffening is provided to prevent local distortions. If the connections can be regarded as behaving elastically, the stiffness and carry-over factors can be modified to allow for the movements which take place in the connections.

For the member AB the slope-deflection Eqs. (3.3) are:

Flexible connections

$$M_a = k(s\theta_a + sc\theta_b), \quad M_b = k(s\theta_b + sc\theta_a), \quad \text{where} \quad k = \frac{EI}{L}. \quad (3.3)$$



The end A is connected to the adjacent structure A' , which rotates through an angle θ'_a . The connection between A and A' has an elastic stiffness G_a such that

$$M_a = G_a(\theta'_a - \theta_a).$$

It will be convenient to write $G_a = \xi_a k$, where ξ_a is a nondimensional measure of the joint stiffness. Similarly, the stiffness of joint B is $G_b = \xi_b k$. Hence

$$M_a = \xi_a k(\theta'_a - \theta_a), \quad M_b = \xi_b k(\theta'_b - \theta_b). \quad (3.64)$$

Elimination of θ_a and θ_b from Eqs. (3.3) and (3.64) leads to the following.

$$M_a = k(s'_a \theta'_a + (sc)'\theta'_b), \quad M_b = k(s'_b \theta'_b + (sc)'\theta'_a), \quad (3.65)$$

where

$$s'_a = \left(s + \frac{s^2(1-c^2)}{\xi_b} \right) / p, \\ s'_b = \left(s + \frac{s^2(1-c^2)}{\xi_a} \right) / p, \\ (sc)' = sc/p, \quad (3.66)$$

$$p = 1 + s \left(\frac{1}{\xi_a} + \frac{1}{\xi_b} \right) + \frac{s^2(1-c^2)}{\xi_a \xi_b}.$$

Equations (3.65) are the modified slope-deflection equations which relate the end-couples M_a , M_b , to the rotations of the nodes at A' and B' . They may be used in any framework analysis and they automatically allow for the effects of the 'slippages', $\theta'_a - \theta_a$ and $\theta'_b - \theta_b$.

3.8.2 Rigid arms at the ends of a member

The member AB in Fig. 3.37(a) is infinitely stiff over a short distance at each end, but has a flexural rigidity EI over the long central portion. This may be taken to represent a member of a framework which is welded to adjacent members of substantial width and with ample local stiffening. The stiffness and carry-over factors may be modified to allow for the rigidity of the ends of the strut.

The overall length AB is L ; the lengths of the rigid portions are a and b ; and the length of the central portion ($A'B'$, of flexural rigidity EI) is L' . The slope-deflection equations, Eqs. (3.18) and (3.22), define the relationships between the moments, forces, and displacements at the ends of $A'B'$:

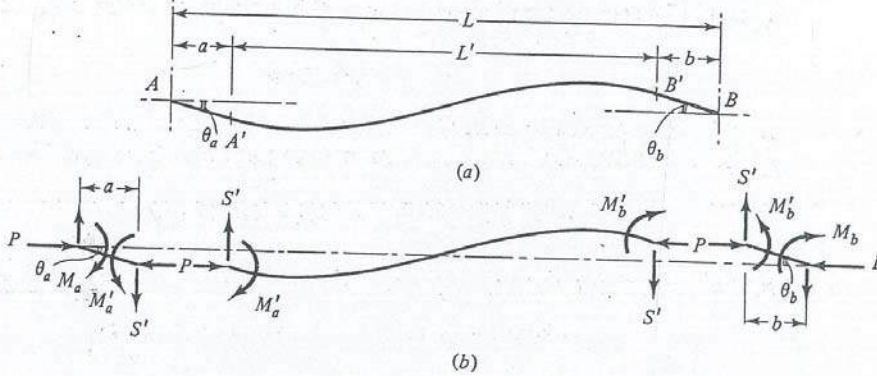


Fig. 3.37 A strut AB with rigid segments AA' and BB' .

$$\begin{aligned} M'_a &= k' \left(s\theta_a + sc\theta_b - s(1+c) \frac{\Delta}{L'} \right), \\ M'_b &= k' \left(s\theta_b + sc\theta_a - s(1+c) \frac{\Delta}{L'} \right), \\ S' &= -s(1+c) \frac{k'}{L'} (\theta_a + \theta_b) + \frac{k'}{L'} \left(2s(1+c) - \frac{\pi^2 P}{P'_E} \right) \frac{\Delta}{L'}, \end{aligned} \quad (3.67)$$

where

$$k' = \frac{EI}{L'}, \quad P'_E = \frac{\pi^2 EI}{L'^2}.$$

If the nodes A and B are to remain at the same level, then

$$a\theta_a + \Delta + b\theta_b = 0. \quad (3.68)$$

The rigid arm at the left-hand end is subjected to the forces and moments shown in Fig. 3.37(b). These include the axial force P , the shear force S' and the bending moment M'_a (both as defined above, but reversed in direction), and finally the moment M_a which is applied at the node A . For equilibrium,

$$M_a = M'_a - aS' - Pa\theta_a. \quad (3.69a)$$

Similarly, at the right-hand end,

$$M_b = M'_b - bS' - Pb\theta_b. \quad (3.69b)$$

Equation (3.68) may be used to eliminate Δ from (3.67). Then M'_a and M'_b may be eliminated by the use of Eqs. (3.69). The result may be expressed as follows.

$$M_a = k(s'_a\theta_a + (sc)'\theta_b), \quad M_b = k(s'_b\theta_b + (sc)'\theta_a). \quad (3.70)$$

where

$$\begin{aligned} s'_a &= \frac{L}{E} \left\{ s + \alpha(1 + \alpha) \left[2s(1 + c) - \frac{PL^2}{EI} \right] \right\}, \\ s'_b &= \frac{L}{E} \left\{ s + \beta(1 + \beta) \left[2s(1 + c) - \frac{PL^2}{EI} \right] \right\}, \\ (sc)' &= \frac{L}{E} \left\{ sc + s(1 + c)(\alpha + \alpha\beta + \beta) - \frac{PL^2}{EI} \alpha\beta \right\}, \end{aligned} \quad (3.71)$$

and

$$k = \frac{EI}{L}, \quad \alpha = \frac{a}{L}, \quad \beta = \frac{b}{L}.$$

Equations (3.70) are the modified slope-deflection equations which relate the end-couples M_a , M_b , to the rotations of the nodes A and B when the short regions at the ends of the member are rigid. They may be used in any framework analysis and they automatically allow for the nonuniformity of the cross-section of the strut.

The effect of inserting elastic connections at A' and B' can be allowed for by modifying the stiffness and carry-over factors in two stages. First the factors for the length $A'B'$, complete with elastic connections at the ends, are modified by using Eqs. (3.66). Then these modified factors are inserted in Eqs. (3.71) to obtain the final factors for AB . It will be found that the equations can be used without modification only if the connections at A' and B' have the same stiffness.

3.9 MATRIX ANALYSIS OF FRAMEWORKS

$$M_{ab} = Sk(\theta_a + C\theta_b - (1 + C)\frac{\Delta}{l})$$

$$M_{ba} = Sk(C\theta_a + \theta_b - (1 + C)\frac{\Delta}{l})$$

$$V = \frac{Sk(1 + C)}{l}(\theta_a + \theta_b) - \left(\frac{2Sk(1 + C)}{l^2} - \frac{P}{l} \right) \Delta$$

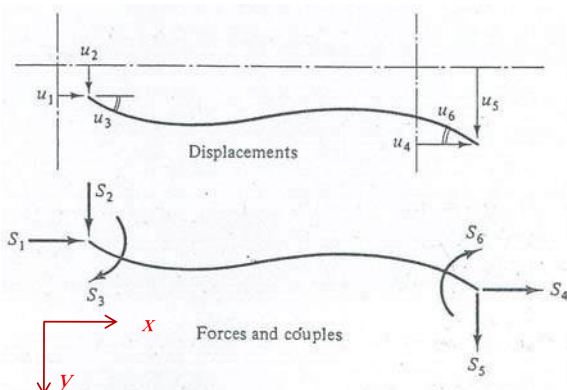
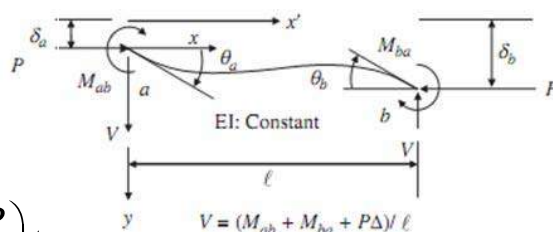


Fig. 1.22 Notation for displacements (u) and forces or couples (S).

$$\begin{aligned} S_1 &= +P \\ S_2 &= V \\ S_3 &= -V \\ S_4 &= -P \end{aligned}$$

$$\begin{aligned} u_1 &= \text{not defined,} \\ u_2 &= \text{not defined,} \\ u_5 &= u_2 + \Delta \\ u_4 &= +u_1 - \frac{PL}{EA} \end{aligned}$$

$$\begin{aligned} S_3 &= +M_a & u_3 &= +\theta_a, \\ S_6 &= +M_b & u_6 &= +\theta_b. \\ u_5 - u_2 &= \Delta, \\ u_1 - u_4 &= \frac{PL}{EA}, \end{aligned} \quad (3.7)$$

Member Force-Deformation Relation in Local Coordinates

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{2}{l^2} Sk(1+C) - \frac{P}{l} & \frac{Sk(1+C)}{l} & 0 & -\frac{2}{l^2} Sk(1+C) + \frac{P}{l} & \frac{Sk(1+C)}{l} \\ 0 & \frac{Sk(1+C)}{l} & Sk & 0 & -\frac{Sk(1+C)}{l} & SkC \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{2}{l^2} Sk(1+C) + \frac{P}{l} & -\frac{Sk(1+C)}{l} & 0 & \frac{2}{l^2} Sk(1+C) - \frac{P}{l} & -\frac{Sk(1+C)}{l} \\ 0 & \frac{Sk(1+C)}{l} & SkC & 0 & -\frac{Sk(1+C)}{l} & Sk \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{Bmatrix} = \begin{bmatrix} p & 0 & 0 & -p & 0 & 0 \\ 0 & q & r & 0 & -q & r \\ 0 & r & U & 0 & -r & V \\ -p & 0 & 0 & p & 0 & 0 \\ 0 & -q & -r & 0 & q & -r \\ 0 & r & V & 0 & -r & U \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$U = Sk \quad V = SkC \quad p = \frac{EA}{l}$$

$$q = \frac{2}{l^2}(U + V) - \frac{P}{l} \quad r = \frac{U + V}{l}$$

$$q = \frac{EI}{L^3} \theta^2 \left\{ \frac{\tan(\theta/2)}{\tan(\theta/2) - (\theta/2)} - 1 \right\},$$

$$r = \frac{EI}{L^2} \frac{\theta^2}{2} \left\{ \frac{\tan(\theta/2)}{\tan(\theta/2) - (\theta/2)} \right\},$$

$$\theta = \pi \sqrt{\frac{P}{P_E}}.$$

$$U = \frac{EI}{L} \frac{\theta}{2} \left\{ \frac{1 - \theta \cot \theta}{\tan(\theta/2) - (\theta/2)} \right\}, \quad V = \frac{EI}{L} \frac{\theta}{2} \left\{ \frac{\theta \operatorname{cosec} \theta - 1}{\tan(\theta/2) - (\theta/2)} \right\},$$

A stiffness matrix for small axial forces.

The geometric stiffness matrix

$$U \rightarrow \frac{4EI}{L} - \frac{2}{15}PL,$$

$$r \rightarrow \frac{6EI}{L^2} - \frac{P}{10},$$

$$V \rightarrow \frac{2EI}{L} + \frac{PL}{30},$$

$$q \rightarrow \frac{12EI}{L^3} - \frac{6P}{5L}.$$

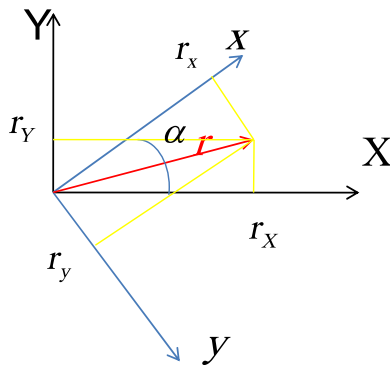
$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12L & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$-\frac{P}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (3.77a)$$

or

$$S = (k_E + k_G)u. \quad (3.77b)$$

Transformation Matrix

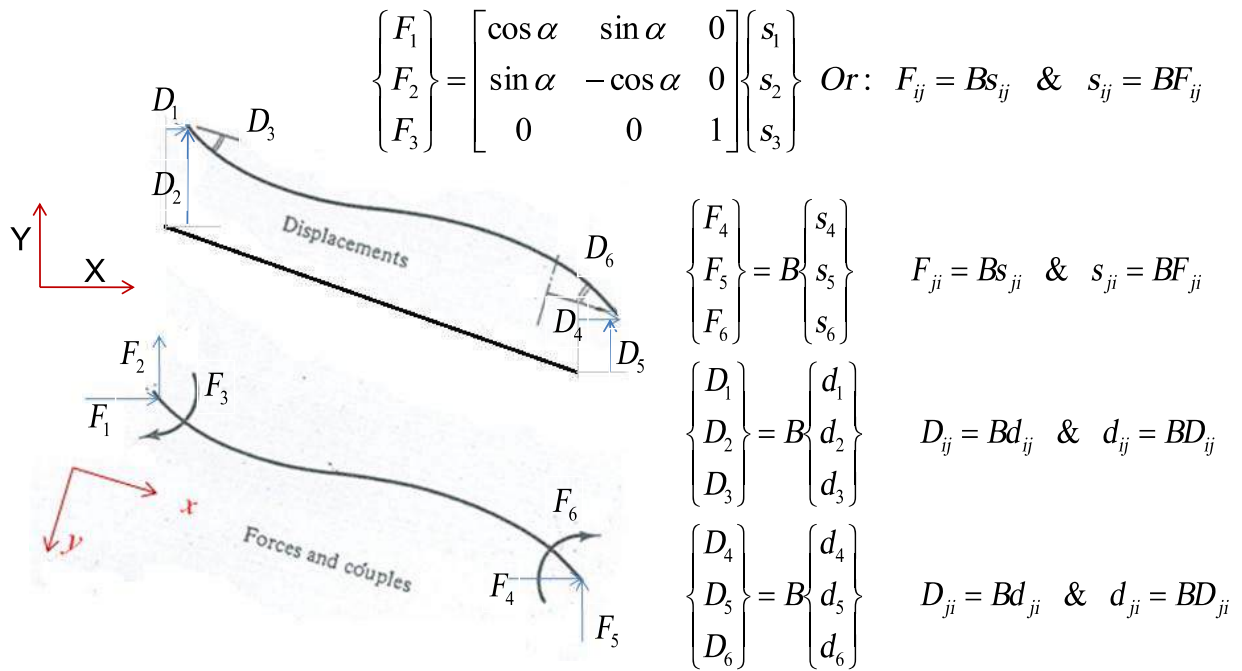


$$r_X = r_x \cos \alpha + r_y \sin \alpha$$

$$r_Y = r_x \sin \alpha - r_y \cos \alpha$$

$$\begin{Bmatrix} r_X \\ r_Y \end{Bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix} \begin{Bmatrix} r_x \\ r_y \end{Bmatrix}$$

Member Force-Deformation Relation in Global Coordinates



$$\{S\} = [k]\{d\}$$

$$\begin{Bmatrix} s_{ij} \\ s_{ji} \end{Bmatrix} = \begin{bmatrix} k_{ii}^j & k_{ij} \\ k_{ji} & k_{jj}^i \end{bmatrix} \begin{Bmatrix} d_{ij} \\ d_{ji} \end{Bmatrix} \Rightarrow \begin{aligned} s_{ij} &= k_{ii}^j d_{ij} + k_{ij} d_{ji} \\ s_{ji} &= k_{ji} d_{ij} + k_{jj}^i d_{ji} \end{aligned}$$

$$k_{ii}^j = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} - P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & 1/10 \\ 0 & 1/10 & 2I/15 \end{bmatrix}$$

$$k_{ij} = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & -6EI/L^2 & 2EI/L \end{bmatrix} - P \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6/5I & 1/10 \\ 0 & -1/10 & -I/30 \end{bmatrix}$$

$$k_{ji} = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L \end{bmatrix} - P \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6/5I & -1/10 \\ 0 & 1/10 & -I/30 \end{bmatrix}$$

$$k_{jj}^i = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix}$$

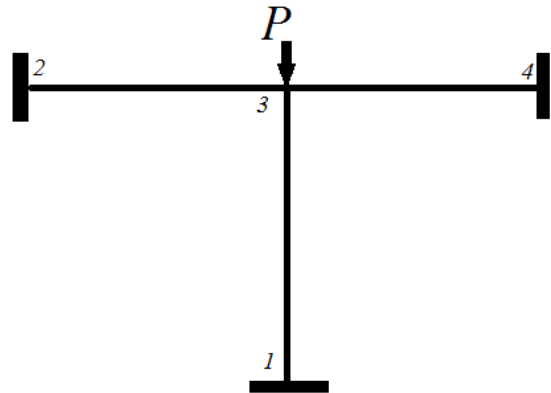
$$\begin{aligned} \Rightarrow B s_{ij} &= B k_{ii}^j B D_{ij} + B k_{ij} B D_{ji} \Rightarrow F_{ij} = K_{ii}^j D_{ij} + K_{ij} D_{ji} \\ \Rightarrow B s_{ji} &= B k_{ji} B D_{ij} + B k_{jj}^i B D_{ji} \Rightarrow F_{ji} = K_{ji} D_{ij} + K_{jj}^i D_{ji} \\ \begin{Bmatrix} F_{ij} \\ F_{ji} \end{Bmatrix} &= \begin{bmatrix} K_{ii}^j & K_{ij} \\ K_{ji} & K_{jj}^i \end{bmatrix} \begin{Bmatrix} D_{ij} \\ D_{ji} \end{Bmatrix} \quad \text{Where:} \quad \begin{aligned} K_{ii}^j &= B k_{ii}^j B & K_{ij} &= B k_{ij} B \\ K_{ji} &= B k_{ji} B & K_{jj}^i &= B k_{jj}^i B \end{aligned} \end{aligned}$$

مثال: با استفاده از روش ماتریسی بار بحرانی قاب شکل زیر را بدست آورید.

$$\frac{EA}{L} = \frac{24EI}{L^3} \quad \text{برای تمامی اعضا یکسان است.} \quad E, I, L, A$$

نمایش کیفی ماتریس سختی سازه

$$\begin{bmatrix} K_{11}^3 & 0 & K_{13} & 0 \\ 0 & K_{22}^3 & K_{23} & 0 \\ K_{31} & K_{32} & K_{33}^2 + K_{33}^3 + K_{33}^4 & K_{34} \\ 0 & 0 & K_{43} & K_{44}^3 \end{bmatrix}$$



نمایش کیفی ماتریس سختی سازه
پس از اعمال شرایط مرزی

$$\left[K_{33}^2 + K_{33}^3 + K_{33}^4 \right]$$

محاسبه ماتریس سختی اعضا:

عضو ۱۳:

$$\alpha = 90 \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

محاسبه ماتریس سختی عضو ۱۳
در دستگاه مختصات محلی:

$$K_{33}^1 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - \alpha P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix} \right)$$

$$K_{33}^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - \alpha P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{33}^1 = \left(\begin{bmatrix} 12EI/L^3 & 0 & -6EI/L^2 \\ 0 & EA/L & 0 \\ -6EI/L^2 & 0 & 4EI/L \end{bmatrix} - \alpha P \begin{bmatrix} 6/5I & 0 & -1/10 \\ 0 & 0 & 0 \\ -1/10 & 0 & 2I/15 \end{bmatrix} \right)$$

عضو ۱۳:

$$\alpha = 0 \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

محاسبه ماتریس سختی عضو ۲۳
در دستگاه مختصات محلی:

$$k_{33}^2 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \right)$$

محاسبه ماتریس سختی عضو ۲۳
در دستگاه مختصات عمومی:

$$K_{33}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{33}^2 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} \right)$$

عضو ۱۳:

$$\alpha = 0 \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

محاسبه ماتریس سختی عضو ۳۴
در دستگاه مختصات محلی:

$$k_{33}^4 = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix}$$

محاسبه ماتریس سختی عضو ۳۴
در دستگاه مختصات عمومی:

$$K_{33}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{33}^1 = \begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

$$K = \begin{bmatrix} 60EI/L^3 & 0 & -6EI/L^2 \\ 0 & 48EI/L^3 & 0 \\ -6EI/L^2 & 0 & 12EI/L \end{bmatrix} - \alpha P \begin{bmatrix} 6/5L & 0 & -1/10 \\ 0 & 0 & 0 \\ -1/10 & 0 & 2L/15 \end{bmatrix} \begin{Bmatrix} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 60EI/L^3 - 6\alpha P/5L & 0 & -6EI/L^2 + \alpha P/10 \\ 0 & 48EI/L^3 & 0 \\ -6EI/L^2 + \alpha P/10 & 0 & 12EI/L - 2\alpha PL/15 \end{vmatrix} = 0$$

$$\alpha = 1/2$$

$$(60 - 3\beta/5)(12 - \beta/15) - (-6 + \beta/20)^2 = 0$$

$$720 + \beta^2/25 - 11.2\alpha - (36 + \beta^2/400 - 3\beta/5)$$

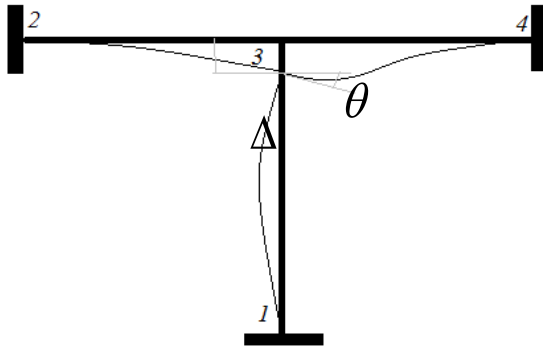
$$3\beta^2/80 - 10.6\beta + 684 = 0$$

$$3\beta^2 - 848\beta + 54720 = 0$$

$$\beta = 99.67$$

$$P_{cr} = 99.67 \frac{EI}{L^2}$$

حل مسئله به روش شیب-افت
با صرف نظر از تغییر شکل های محوری تیرها



$$M_{31} = sk\theta$$

$$M_{32} = 4k(\theta - 1.5\Delta / L)$$

$$M_{34} = 4k(\theta + 1.5\Delta / L)$$

$$M_{31} + M_{32} + M_{34} = sk\theta + 8k\theta = 0 \Rightarrow s = -8$$

$$V_{1cr} = 3.26 \frac{\pi^2 EI}{L^2}$$

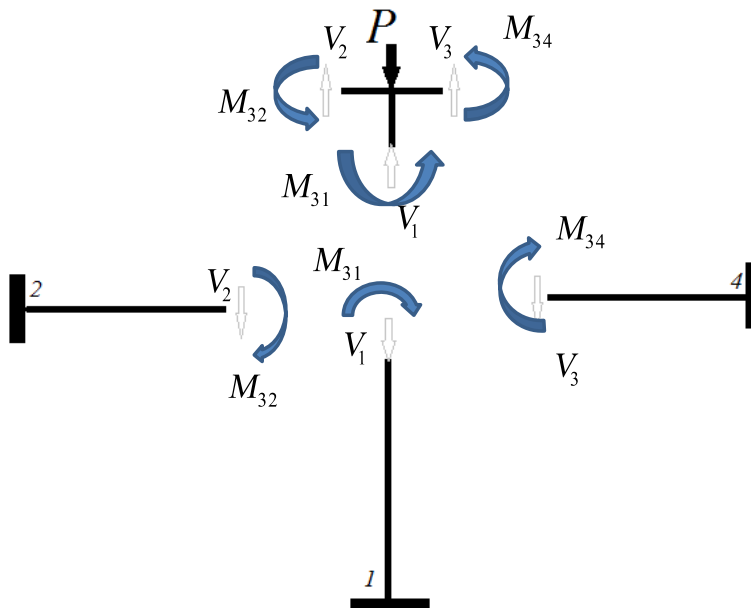
$$V_2 = -(M_{32} + M_{23}) / L = -4k(1.5\theta - 3\Delta / L)$$

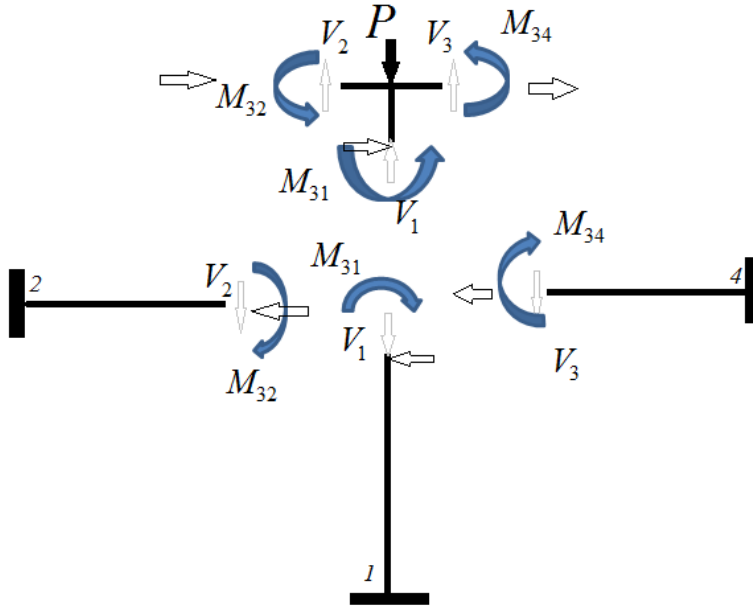
$$V_3 = (M_{34} + M_{43}) / L = 4k(1.5\theta + 3\Delta / L)$$

$$V_1 = \frac{AE}{L} \Delta$$

$$V_1 + V_2 + V_3 = 24k\Delta / L + \frac{AE}{L} \Delta = P \Rightarrow V_1 = \frac{P}{2}$$

$$P_{cr} = 6.52 \frac{\pi^2 EI}{L^2}$$





$$M_{31} = sk(\theta - (1+c)\Delta_2 / L)$$

$$M_{32} = 4k(\theta - 1.5\Delta_1 / L)$$

$$M_{34} = 4k(\theta + 1.5\Delta_1 / L)$$

$$M_{31} + M_{32} + M_{34} = sk(\theta - (1+c)\Delta_2 / L) + 8k\theta = 0$$

$$p_1 + p_2 + p_3 = 0$$

$$p_1 = -AE\Delta_2 / L \quad p_2 = -AE\Delta_2 / L$$

$$p_3 = (sk(1+c)(\theta - 2\Delta_2 / L) + V_1\Delta_2) / L$$

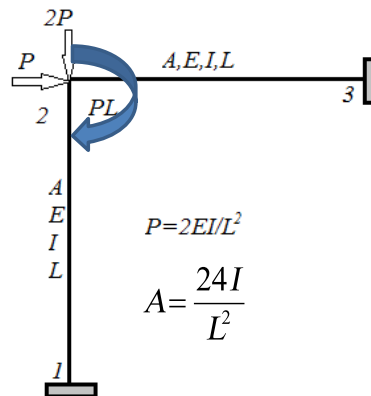
$$-2AE\Delta_2 / L + sk(1+c)(\theta - 2\Delta_2 / L) / L + V_1\Delta_2 / L = 0$$

$$(s+8)\theta - s(1+c)\Delta_2 / L = 0$$

$$s(1+c)\theta / L - (2s(1+c) + 48 - \mu^2 L^2)\Delta_2 / L^2 = 0$$

مثال: قاب شکل زیر را تحلیل نمایید.

نمایش کیفی ماتریس سختی کل سازه پس از اعمال شرایط مرزی:



$$[K_{22}^1 + K_{22}^3]$$

عضو ۱۲:

$$\alpha = 90 \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

محاسبه ماتریس سختی عضو ۱۲ در دستگاه مختصات محلی:

$$K_{22}^1 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - P_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix} \right)$$

محاسبه ماتریس سختی عضو ۱۲ در دستگاه مختصات عمومی:

$$K_{22}^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - P_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{22}^1 = \left(\begin{bmatrix} 12EI/L^3 & 0 & -6EI/L^2 \\ 0 & EA/L & 0 \\ -6EI/L^2 & 0 & 4EI/L \end{bmatrix} - P_1 \begin{bmatrix} 6/5I & 0 & -1/10 \\ 0 & 0 & 0 \\ -1/10 & 0 & 2I/15 \end{bmatrix} \right)$$

محاسبه ماتریس سختی عضو ۲۳ در دستگاه مختصات محلی:

$$\alpha = 0 \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K_{22}^3 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} - P_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & 1/10 \\ 0 & 1/10 & 2I/15 \end{bmatrix} \right)$$

محاسبه ماتریس سختی عضو ۲۳ در دستگاه مختصات عمومی:

$$K_{22}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} - P_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & 1/10 \\ 0 & 1/10 & 2I/15 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_{22}^3 = \left(\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{bmatrix} - P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6/5I & -1/10 \\ 0 & -1/10 & 2I/15 \end{bmatrix} \right)$$

$$K = \left[\begin{array}{ccc} 36EI/L^3 & 0 & -6EI/L^2 \\ 0 & 36EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & -6EI/L^2 & 8EI/L \end{array} \right] - \left[\begin{array}{ccc} 6P_1/5L & 0 & -P_1/10 \\ 0 & 6P_2/5L & -P_2/10 \\ -P_1/10 & -P_2/10 & 2(P_1+P_2)L/15 \end{array} \right]$$

$$[K] \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \\ PL \end{Bmatrix} \quad \text{مرحله اول حل خطی:}$$

$$\left(\left[\begin{array}{ccc} 36EI/L^3 & 0 & -6EI/L^2 \\ 0 & 36EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & -6EI/L^2 & 8EI/L \end{array} \right] \right) \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \\ PL \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} \frac{11PL^3}{216EI} \\ \frac{216EI}{-7PL^3} \\ \frac{216EI}{30PL^2} \\ \frac{216EI}{216EI} \end{Bmatrix}$$

$$F_{12} = K_{11}^2 D_{12} + K_{12} D_{21}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}^{12} = \begin{Bmatrix} -12EI/L^3 & 0 & 6EI/L^2 \\ 0 & EA/L & 0 \\ -6EI/L^2 & 0 & 4EI/L \end{Bmatrix} \begin{Bmatrix} \frac{11PL^3}{216EI} \\ \frac{216EI}{-7PL^3} \\ \frac{216EI}{30PL^2} \\ \frac{216EI}{216EI} \end{Bmatrix} = \begin{Bmatrix} \frac{2P}{9} \\ \frac{7P}{9} \\ \frac{9}{PL} \\ \frac{35}{35} \end{Bmatrix} \Rightarrow P_1 = \frac{7P}{9} = 1.556 \frac{EI}{L^2}$$

$$F_{23} = K_{22}^3 D_{23} + K_{23} D_{32}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}^{23} = \begin{Bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & 4EI/L \end{Bmatrix} \begin{Bmatrix} \frac{11PL^3}{216EI} \\ \frac{216EI}{-7PL^3} \\ \frac{216EI}{30PL^2} \\ \frac{216EI}{216EI} \end{Bmatrix} = \begin{Bmatrix} \frac{11P}{9} \\ -\frac{11P}{9} \\ \frac{3PL}{4} \end{Bmatrix} \Rightarrow P_1 = \frac{11P}{9} = 2.444 \frac{EI}{L^2}$$

مرحله دوم حل غیر خطی:
گام اول:

$$\left(\left[\begin{array}{ccc} 36EI/L^3 & 0 & -6EI/L^2 \\ 0 & 36EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & -6EI/L^2 & 8EI/L \end{array} \right] - \left[\begin{array}{ccc} 84EI/45L^3 & 0 & -14EI/90L^2 \\ 0 & 132EI/45L^3 & -22EI/90L^2 \\ -14EI/90L^2 & -22EI/10L^2 & 72EI/135L \end{array} \right] \right) \begin{Bmatrix} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \\ PL \end{Bmatrix}$$

$$\left(\left[\begin{array}{ccc} 512EI/15L^3 & 0 & -526EI/90L^2 \\ 0 & 496EI/15L^3 & -518EI/90L^2 \\ -526EI/90L^2 & -518EI/90L^2 & 112EI/15L \end{array} \right] \right) \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \\ PL \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Delta_{2x} \\ \Delta_{2y} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.0551 \frac{PL^3}{EI} \\ -0.0344 \frac{PL^3}{EI} \\ 0.1506 \frac{PL^2}{EI} \end{Bmatrix}$$

$$F_{12} = K_{11}^2 D_{12} + K_{12} D_{21} \Rightarrow P_1 = 0.8256 P = 1.652 \frac{EI}{L^2}$$

$$F_{23} = K_{22}^3 D_{23} + K_{23} D_{32} \Rightarrow P_2 = 1.3224 P = 2.6448 \frac{EI}{L^2}$$

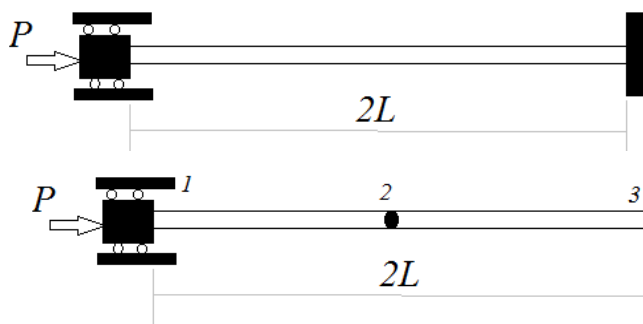
حل غیر خطی:

گام دوم:

$$\left[\begin{array}{ccc} 30.174EI/L^3 & 0 & -5.8348EI/L^2 \\ 0 & 32.8286EI/L^3 & -5.7355EI/L^2 \\ -5.8348EI/L^2 & -5.7355EI/L^2 & 7.4271EI/L \end{array} \right] \begin{Bmatrix} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \\ PL \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Delta_{1x} \\ \Delta_{1y} \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0.05539 \frac{PL^3}{EI} \\ -0.034445 \frac{PL^3}{EI} \\ 0.1516 \frac{PL^2}{EI} \end{Bmatrix}$$

$$F_{12} = K_{11}^2 D_{12} + K_{12} D_{21} \Rightarrow P_1 = 0.8267 P = 1.6534 \frac{EI}{L^2}$$

$$F_{23} = K_{22}^3 D_{23} + K_{23} D_{32} \Rightarrow P_2 = 1.32936 P = 2.65872 \frac{EI}{L^2}$$



بار بحرانی ستون شکل زیر را بدست آورید:

ماتریس سختی سوار شده قبل از اعمال شرایط مرزی:

$$\begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 & 0 & 0 & 0 \\ 0 & 12EI/L^3 - 6P/5l & -6EI/L + P/10^2 & 0 & -12EI/L^3 + 6P/5l & -6EI/L^2 + P/10 & 0 & 0 & 0 \\ 0 & -6EI/L^2 + P/10 & 4EI/L - 2Pl/15 & 0 & 6EI/L^2 - P/10 & 2EI/L + Pl/30 & 0 & 0 & 0 \\ -EA/L & 0 & 0 & 2EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & -12EI/L^3 + 6P/5l & 6EI/L^2 - P/10 & 0 & 24EI/L^3 - 12P/5l & 0 & 0 & -12EI/L^3 + 6P/5l & -6EI/L^2 + P/10 \\ 0 & -6EI/L^2 + P/10 & 2EI/L + Pl/30 & 0 & 0 & 8EI/L - 4Pl/15 & 0 & 6EI/L^2 - P/10 & 2EI/L + Pl/30 \\ 0 & 0 & 0 & -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & 0 & 0 & 0 & -12EI/L^3 + 6P/5l & 6EI/L^2 - P/10 & 0 & 12EI/L^3 - 6P/5l & 6EI/L - P/10^2 \\ 0 & 0 & 0 & 0 & -6EI/L^2 + P/10 & 2EI/L + Pl/30 & 0 & 6EI/L^2 - P/10 & 4EI/L - 2Pl/15 \end{bmatrix}$$

ماتریس سختی سوار شده پس از اعمال شرایط مرزی:

$$\begin{bmatrix} EA/L & -EA/L & 0 & 0 \\ -EA/L & 2EA/L & 0 & 0 \\ 0 & 0 & 24EI/L^3 - 12P/5I & 0 \\ 0 & 0 & 0 & 8EI/L - 4Pl/15 \end{bmatrix}$$
$$P_{cr1} = \frac{10EI}{L^2} \quad P_{cr2} = \frac{30EI}{L^2}$$