Solutions Manual
To Accompany

R.W. Clough and J. Penzien

DYNAMICS OF STRUCTURES

Second Edition

Francisco Medina
University of Puerto Rico, Mayaguez

McGraw-Hill, Inc.

New York St. Louis San Francisco Auckland Bogotá Caracas Lisbon
London Madrid Mexico City Milan Montreal New Delhi
San Juan Singapore Sydney Tokyo Toronto

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ISBN 0-07-011395-5

1 2 3 4 5 6 7 8 9 0 BKM BKM 9 0 9 8 7 6 5

PREFACE

This Solutions Manual contains the solution to all the problems proposed in the text Dynamics of Structures. To solve many of these problems, the student should be very familiar with trigonometric and hyperbolic relationships, integral and differential calculus, matrix algebra, and matrix structural analysis. The student also has to have some knowledge of differential equations, and probability and theory of residues (these last two topics for Part IV).

The problems were solved from the first edition of the book, systematically and for no particular reason, when I was pursuing graduate studies at Berkeley in the late '70s, without having in mind that these solutions would later be assembled in a Solutions Manual. Therefore, the way the solutions are carried out may not be optimally didactic or elegant; however, they present the advantage of showing a step-by-step procedure that a regular student would perform in attempting to solve these problems in a homework or test framework, or simply just for fun.

All the problems can be solved by hand, but in a small number of cases it was preferred to present a solution carried out with the help of a computer program or a programmable calculator. Some of the problems may be solved differently (especially in Part IV); however, the answers are correct, and almost all have been independently checked.

In order to prepare the Solutions Manual, the existing solutions for the first edition of the book were revised to conform to the new edition, and then transcribed into the manual's current format. This procedure was checked carefully. Nevertheless, if the reader notices any error, I would be pleased to be advised.

Finally, I wish to thank the book's authors for allowing me to have an enjoyable time while studying in Berkeley. The transcription of the solutions was patiently done by Mr. José Orozco, who deserves to be given all the credit for it.

Francisco Medina
Mayagüez, December 1994

Problem 2-2

Eq. (2-24): $\omega^2 = \frac{k}{m} \longrightarrow \omega = \sqrt{\frac{40}{2}} = 4.47 \text{ 1/sec.}$

Eq. (2-49):

$$v(t) = e^{-\xi \omega t}\left[\left(\frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D}\right)\sin \omega_D t + v(0)\cos \omega_D t\right]$$

$$\therefore \dot{v}(t) = e^{-\xi\omega t}\left(\left[-\xi\omega \frac{\dot{v}(0) + v(0)\xi\omega}{\omega_D} - \omega_D v(0)\right]\sin \omega_D t + \right.$$

$$\left.\left\{\left[\dot{v}(0) + \cancel{v(0)\xi\omega}\right] - \cancel{\xi\omega v(0)}\right\}\cos \omega_D t\right)$$

$$\dot{v}(t) = e^{-\xi\omega t}\left\{\dot{v}(0)\cos \omega_D t - \frac{\xi\omega\dot{v}(0) + \left[(\xi\omega)^2 + \omega_D^2\right]v(0)}{\omega_D}\sin \omega_D t\right\}$$

Eq. (2-46):

$$\omega_D = \omega\sqrt{1-\xi^2} \longrightarrow \dot{v}(t) = e^{-\xi\omega t}\left\{\dot{v}(0)\cos\omega_D t - \left[\xi\dot{v}(0) + \omega v(0)\right]\frac{\omega}{\omega_D}\sin \omega_D t\right\}$$

$$\dot{v}(t) = e^{-\xi\omega t}\left\{\dot{v}(0)\cos\omega_D t - \frac{\xi\dot{v}(0) + \omega v(0)}{\sqrt{1-\xi^2}}\sin \omega_D t\right\}$$

Eq. (2-44): $\xi = \frac{c}{2m\omega} = \frac{c}{2(2)(4.47)} = 0.0559\,c$

(a) $c = 0 \rightarrow \xi = 0 \rightarrow \omega_D = \omega$

$$\therefore v(t=1) = \frac{5.6}{4.47}\sin 4.47 + 0.7\cos 4.47 = -1.38 \text{ in}$$

$$\dot{v}(t=1) = 5.6\cos 4.47 - 4.47(0.7)\sin 4.47 = 1.69 \text{ in/sec.}$$

▶ $\boxed{v(1) = -1.4 \text{ in}\ , \quad \dot{v}(1) = 1.7 \text{ in/sec.}}$

(continued on following page)

Problem 2-1

(a) From first eq. of E2-1:

$$T = 2\pi \sqrt{\frac{W}{gk}}$$

$$k = \left(\frac{2\pi}{T}\right)^2 \frac{W}{g}$$

Since $v_{max}(t) = v(kT_D)$,

where $k = 0, 1, \ldots$

$T_D = 0.64$ sec if ξ small, $T_D \doteq T$

$$\therefore k = \left(\frac{2\pi}{0.64 \text{ sec}}\right)^2 \frac{200 \text{ kips}}{386 \text{ in/sec}^2} = 49.9 \text{ kips/in}$$

$$\blacktriangleright \boxed{k = 50 \text{ kips/in}}$$

(b) Eq. (2-54):

$$\delta \equiv \ln \frac{v_n}{v_{n+1}} = \ln \frac{v_0}{v_1}$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \rightarrow \xi = \frac{1}{\sqrt{\left(\frac{2\pi}{\delta}\right)^2 + 1}}$$

Since $\delta = \ln \frac{1.20}{0.86} = 0.333$

$$\xi = \frac{1}{\sqrt{\left(\frac{2\pi}{0.333}\right)^2 + 1}} = 0.0529$$

Directly from eq. (2-55):

$$\delta \doteq 2\pi\xi \longrightarrow \xi = \frac{0.333}{2\pi} = 0.0530$$

$$\blacktriangleright \boxed{\xi = 5.3\%}$$

(a') a. revisited

Eq. (2-46):

$\omega_D = \omega\sqrt{1-\xi^2}$,

by eq. (2-35): $T = \frac{2\pi}{\omega}$

$$T = T_D\sqrt{1-\xi^2}$$

$$k = \frac{49.9}{1-\xi^2} = 50.0 \text{ kips/in}$$

(c) Eq. (2-44): $\xi = \frac{c}{2m\omega}$

$$m = \frac{W}{g}$$

Eq. (2-35): $T = \frac{2\pi}{\omega}$

Therefore $c = \frac{4\pi}{T} \cdot \frac{W}{g}\xi$

From part (a'): $T = T_D\sqrt{1-\xi^2}$

$$c = \frac{4\pi}{T_D} \cdot \frac{\xi}{\sqrt{1-\xi^2}} \cdot \frac{W}{g}$$

$$\therefore c = \frac{4\pi}{0.64 \text{ sec}} \cdot \frac{0.0529}{\sqrt{1-(0.0529)^2}} \cdot \frac{200 \text{ kips}}{386 \text{ in/sec}^2}$$

$$c = 0.539 \text{ kips}\cdot\text{sec/in}$$

if it is assumed $T = T_D$,

$$c = 0.538 \text{ kips}\cdot\text{sec/in}$$

$$\blacktriangleright \boxed{c = 0.54 \text{ kips}\cdot\text{sec/in}}$$

Problem 2-2 (con'd)

(b) $c = 2.8 \longrightarrow \xi = 0.0559\,(2.8) = 0.157$

$$\omega_D = 4.47\sqrt{1-(0.157)^2} = 4.41 \ {}^{1}/_{sec.}$$

$$\therefore\ v(t=1) = e^{-(0.157)(4.41)}\left[\frac{5.6+0.7\,(0.157)(4.47)}{4.41}\sin 4.41 + 0.7\cos 4.41\right]$$

$$v(t=1) = -0.764 \text{ in}$$

$$\dot v(t=1) = e^{-(0.157)(4.41)}\left[5.6\cos 4.41 - \frac{0.157(5.6)+4.41(0.7)}{\sqrt{1-(0.157)^2}}\sin 4.41\right]$$

$$\dot v(t=1) = 1.10 \text{ in/sec.}$$

▶ $v(1) = -0.76$ in , $\dot v(1) = 1.1$ in/sec.

Problem 2-3

Eq. (2-33) : $v(t) = \dfrac{\dot v(0)}{\omega}\sin\omega t + v(0)\cos\omega t$

Eq. (2-24) : $\omega^2 = \dfrac{k}{m} \longrightarrow \omega = \sqrt{\dfrac{20}{5}} = 2 \ {}^{1}/_{sec.}$

$$\therefore\ v(t=1.2) = \frac{\dot v(0)}{2}\sin 2(1.2) + 1.8\cos 2(1.2) = 1.8$$

$$\dot v(0) = 9.26 \text{ in/sec}$$

(a) $$v(t=2.4) = \frac{9.26}{2}\sin 2(2.4) + 1.8\cos 2(2.4) = -4.45 \text{ in}$$

▶ $v(2.4) = -4.5$ in

(b) Eq. (2-37) : $\rho = \sqrt{[v(0)]^2 + \left[\dfrac{\dot v(0)}{\omega}\right]^2}$

$$\rho = \sqrt{(1.8)^2 + \left(\frac{9.26}{2}\right)^2} = 4.97 \text{ in}$$

▶ $\rho = 5.0$ in

Problem 3-1

Eq. (3-12):
$$R(t)=\left[\frac{1}{1-\beta^2}\right](\sin\bar{\omega}t-\beta\sin\omega t)\ \text{if}\ v(0)=\dot{v}(0)=0$$

From eq. (3-7): $\beta\equiv\dfrac{\bar{\omega}}{\omega}$, $R(\bar{\omega}t)=\left[\dfrac{1}{1-\beta^2}\right]\left(\sin\bar{\omega}t-\beta\sin\dfrac{\bar{\omega}t}{\beta}\right)$

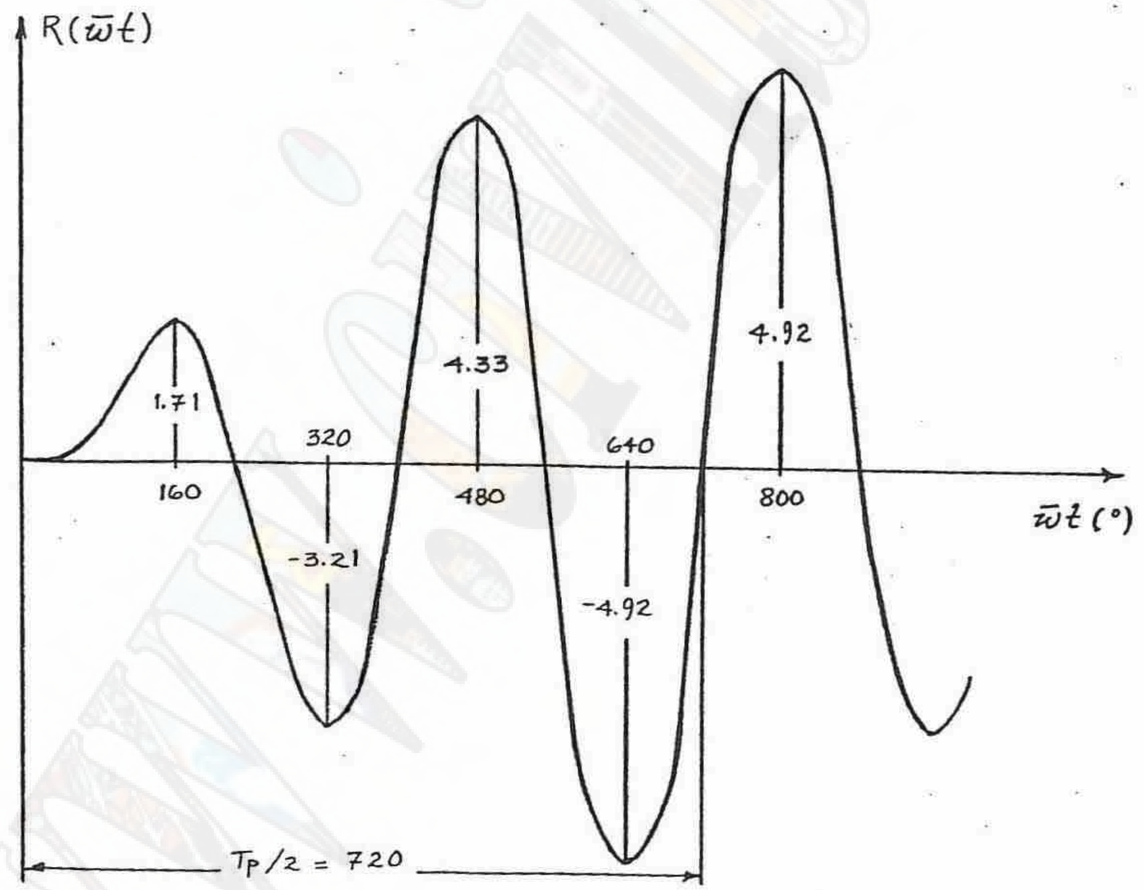
$$\therefore\quad R'(\bar{\omega}t)=\left[\frac{1}{1-\beta^2}\right]\left(\cos\bar{\omega}t-\cos\frac{\bar{\omega}t}{\beta}\right)$$

Therefore, R_{max} when $\bar{\omega}t=2k\pi\pm\dfrac{\bar{\omega}t}{\beta}$, $k=0,1,\ldots$ ($k=0$, only when the sign is "+")

$$\bar{\omega}t=\left(1\mp\frac{1}{\beta}\right)^{-1}2k\pi,\quad k=0,1,\ldots\ (k=0,\text{ only when the sign is "–"})$$

In this case, $\beta=0.8\longrightarrow R(\bar{\omega}t)=2.78\sin\bar{\omega}t-2.22\sin 1.25\,\bar{\omega}t$

and when $\bar{\omega}t=160^\circ k$, $k=0,1,2,\ldots$



$\bar{\omega}t$ (°)	0	80	160	240	320	400	480	560	640	720	800	880
$R(\bar{\omega}t)$	0	.551	1.71	-.485	-3.21	.360	4.33	-.192	-4.92	0	4.92	.192

Problem 3-2

(a) Eq. (3-38) : $R(\omega t) = \frac{1}{2}(\sin \omega t - \omega t \cos \omega t)$

$$R(\omega t = 8\pi) = -\frac{1}{2}(8\pi) = -4\pi$$

$$\boxed{R(8\pi) = -4\pi}$$

From eq. (2-44) : $\xi = \frac{c}{2m\omega}$, and eq. (2-24) : $\omega^2 = \frac{k}{m}$

$$\xi = \frac{c}{2\sqrt{mk}} = \frac{c}{2\sqrt{2(20)}} = \frac{\sqrt{10}}{40}\, c$$

Eq. (3-37) : $R(\omega t) \doteq \frac{1}{2\xi}\left[\left(e^{-\xi\omega t} - 1\right)\cos \omega t + \xi e^{-\xi \omega t} \sin \omega t\right]$

(b) $$R(\omega t = 8\pi) = \frac{1}{2\left(\frac{\sqrt{10}}{40}\, 0.5\right)}\left[e^{-\left(\frac{\sqrt{10}}{40}0.5\right)(8\pi)} - 1\right]\cos 8\pi = -7.97$$

$$\boxed{R(8\pi) = -8.0}$$

(c) $$R(\omega t = 8\pi) = \frac{1}{2\left(\frac{\sqrt{10}}{40}\, 2\right)}\left[e^{-\left(\frac{\sqrt{10}}{40}2\right)(8\pi)} - 1\right]\cos 8\pi = -3.10$$

$$\boxed{R(8\pi) = -3.1}$$

Problem 3-3

(a) At resonance $\beta = 1$ from eq. (3-7) $\beta = \frac{\bar{\omega}}{\omega} \longrightarrow \omega = \bar{\omega} = \omega_p$

Since $T = \frac{2\pi}{\omega}$ eq. (2-35) $\longrightarrow T = T_p$

But $T = 0.572$ sec., third eq. of E3-2 and $T_p = \frac{L}{v_{sp}}$, second eq. of E3-2.

$$\therefore v_{sp} = \frac{L}{T} = \frac{36 ft}{0.572 \text{ sec.}} = 62.94 \text{ ft/sec.} = 42.91 \text{ mph}$$

► $\boxed{v_{sp} = 42.9 \text{ mph}}$

(b) First eq. of E3-2 : $v^t_{max} = v_{g0}\left[\frac{1 + (2\xi\beta)^2}{(1-\beta^2)^2 + (2\xi\beta)^2}\right]^{1/2}$

$$v^t_{max} = 1.2\sqrt{1 + \frac{1}{[2(0.4)]^2}} = 1.921 \text{ in}$$

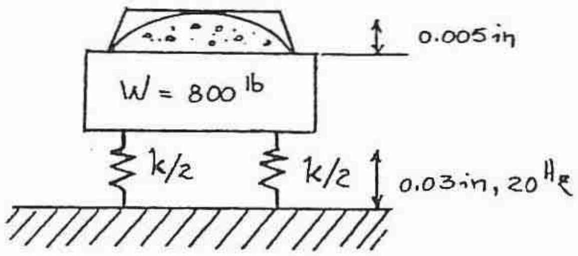
► $\boxed{v^t_{max} = 1.92 \text{ in}}$

(c) $\beta = \frac{\omega_p}{\omega} = \frac{T}{T_p} = \frac{T v_{sp}}{L} = \frac{(0.572 \text{ sec.})(66 \text{ ft/sec.})}{36 \text{ ft}} = 1.049$

$$\therefore v^t_{max} = 1.2\left\{\frac{1 + [2(0.4)(1.049)]^2}{[1 - (1.049)^2]^2 + [2(0.4)(1.049)]^2}\right\}^{1/2} = 1.854 \text{ in}$$

► $\boxed{v^t_{max} = 1.85 \text{ in}}$

Problem 3-4



If $\xi \doteq 0$, eq. (3-49): $TR \doteq \dfrac{1}{\beta^2 - 1}$

where $TR = \dfrac{v^t_{max}}{v_{g0}}$ eq. (3-47)

$\beta^2 = \dfrac{\bar{w}^2 W}{k g}$ from after eq. (3-51)

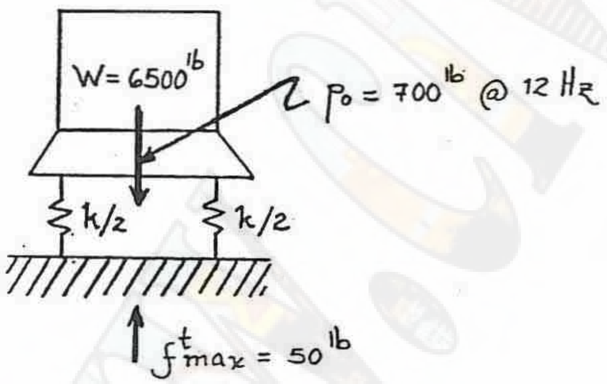
$\bar{w} = 2\pi \bar{f}$

$$\therefore k = \frac{w^2 W}{\beta^2 g} = \frac{1}{\frac{1}{TR} + 1} \cdot \frac{(2\pi\bar{f})^2 W}{g}$$

$$\frac{1}{TR} = \frac{0.03}{0.005} = 6 \longrightarrow k = \frac{1}{6+1} \cdot \frac{[2\pi(20\ {}^{1}/_{sec.})]^2\ 800^{lb}}{32.22\ ft/sec.^2} = 56000\ {}^{lb}/_{ft}$$

$\blacktriangleright \underline{\underline{k = 56\ {}^{kips}/_{ft}}}$

Problem 3-5



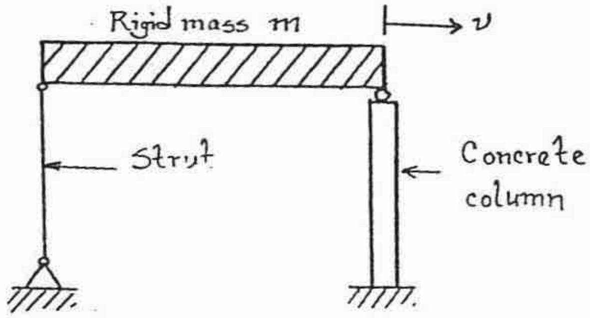
Eq. (3-44): $TR = \dfrac{f_{max}}{p_0}$

From P3-4: $k = \dfrac{1}{\frac{1}{TR}+1} \dfrac{W}{g} (2\pi\bar{f})^2$

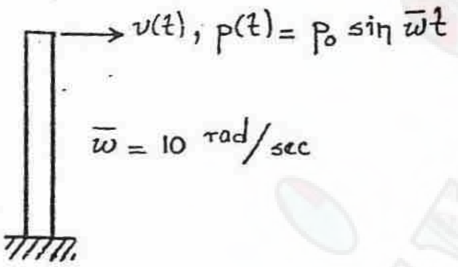
$$\frac{1}{TR} = \frac{700}{50} = 14 \longrightarrow k = \frac{1}{14+1} \cdot \frac{6500^{lb}}{32.22\ ft/sec.^2} \left[2\pi(12)\right]^2 = 76460\ {}^{lb}/_{ft}$$

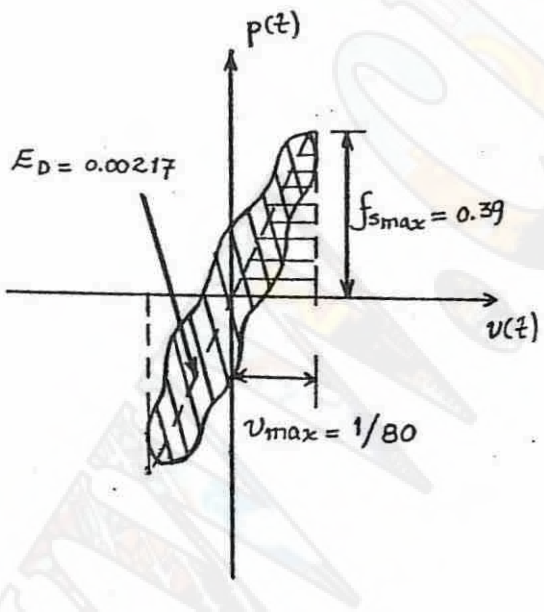
$\blacktriangleright \underline{\underline{k = 76.5\ {}^{kips}/_{ft}}}$

Problem 3-6



Note that in this model only the column provides the stiffness and the energy dissipation of the system for movements in direction v. Hence the model and the column have the same properties.





. Units : kips - ft

(a) From Fig. 3-17: $k\rho = f_{Smax}$

$$k = \frac{f_{s\,max}}{\rho} = \frac{390^{\,lb}}{0.15\,in}$$

$$k = 2600^{\,lb/in} = 31200^{\,lb/ft}$$

$$\blacktriangleright\ \boxed{k = 31^{\,kips/ft}}$$

(b) From eq. (3-66):

$$\xi_{eq.} = \frac{E_D}{2\pi k \rho^2}$$

and Fig. 3-17 :

$$E_S = \frac{1}{2} f_{smax}\, v_{max} = \frac{1}{2} k \rho^2$$

$$\xi_{eq.} = \frac{E_D}{4\pi E_S} = \frac{26\ lb.in}{4\pi(29^{\,lb.in})} = 0.0713$$

Since $\xi = \dfrac{c}{2m\omega}$

$$\xi_{eq.} = \frac{c_{eq.}}{2m\bar{\omega}} \quad \text{(at resonance)}$$

From eq. (3-66): $\xi_{eq.} = \dfrac{E_D}{2\pi m \omega^2 \rho^2}$

$$\therefore c_{eq.} = \frac{E_D}{\pi \bar{\omega} \rho^2} = \frac{26\ lb.in}{\pi(10\ 1/sec.)(0.15\,in)^2}$$

$$c_{eq.} = 36.8\ lb\cdot sec/in = 441\ lb.\,sec/ft$$

$$\blacktriangleright\ \boxed{\xi = 7.1\,\%\ ,\ c = 0.44\ kips\cdot sec/ft}$$

(c) Eq. (3-78): $\zeta = 2\xi\beta$

$$\zeta = 2\xi \quad \text{(at resonance)}$$

$$\zeta = 2(0.0713) = 0.1426$$

$$\blacktriangleright\ \boxed{\zeta = 14.2\,\%}$$

Problem 4-1

$$p(t)\begin{cases} p_0 \sin \dfrac{3\pi}{T_p} t\ , & 0 < t < 2\pi \\ 0 & ,\ 2\pi < t < 3\pi = T_p \end{cases}$$

From eq. (4-1,2):

$$p(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{T_p} t + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{T_p} t \quad , \textit{where}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{T_p}\int_0^{T_p} p(t)dt \\ a_n &= \frac{2}{T_p}\int_0^{T_p} p(t)\cos\frac{2\pi n}{T_p} t\, dt \\ b_n &= \frac{2}{T_p}\int_0^{T_p} p(t)\sin\frac{2\pi n}{T_p} t\, dt \end{aligned}\right\} \text{Eq. } (4\text{-}3)$$

$$\therefore\ a_0 = \frac{1}{T_p}\int_0^{\frac{2}{3}T_p} p_0 \sin\frac{3\pi}{T_p} t\, dt = \frac{P_0}{T_p}\left[\frac{T_p}{3\pi}\left(-\cos\frac{3\pi}{T_p}t\right)\right]_0^{\frac{2}{3}T_p} = 0$$

$$a_n = \frac{2}{T_p}\int_0^{\frac{2}{3}T_p} p_0 \sin\frac{3\pi}{T_p} t\ \cos\frac{2\pi n}{T_p} t\, dt = \frac{2P_0}{T_p}\left[-\frac{\cos(3-2n)\frac{\pi t}{T_p}}{2(3-2n)\frac{\pi}{T_p}} - \frac{\cos(3+2n)\frac{\pi t}{T_p}}{2(3+2n)\frac{\pi}{T_p}}\right]_0^{\frac{2}{3}T_p}$$

$$a_n = \frac{P_0}{\pi}\cdot\frac{-1}{9-4n^2}\left\{(3+2n)\left[\cos(3-2n)\frac{2}{3}\pi - 1\right] + (3-2n)\left[\cos(3+2n)\frac{2}{3}\pi - 1\right]\right\}$$

$$a_n = \frac{P_0}{\pi}\cdot\frac{-1}{9-4n^2}\left[-6 + (3+2n)\cos\frac{4\pi n}{3} + (3-2n)\cos\frac{4\pi n}{3}\right]$$

$$a_n = \frac{6P_0}{\pi}\cdot\frac{1}{9-4n^2}\left(1-\cos\frac{4\pi n}{3}\right) = \frac{6P_0}{\pi}\cdot\frac{1}{9-4n^2}\ 2\sin^2\frac{2\pi n}{3}$$

$$a_n = \frac{12P_0}{\pi}\cdot\frac{1}{9-4n^2}\begin{cases} 0 & ,\ n = 3, 6, \ldots \\ \left(\pm\dfrac{\sqrt{3}}{2}\right)^2 & ,\ n \neq 3, 6, \ldots \end{cases}$$

(continued on following page)

Problem 3-7

(a) $\xi \neq \xi(\omega)$ $\therefore$ From P3-6b: $\xi = 7.1\%$

$$c_{eq.} = \frac{E_D}{\pi \bar{\omega} \rho^2} = \frac{26 \text{ lb.in}}{\pi (20 \text{ 1/sec})(0.15 \text{in})^2}$$

$$c_{eq.} = 18.4 \text{ lb.sec/in} = 221 \text{ lb.sec/ft}$$

$\blacktriangleright$ $\xi = 7.1\%$, $c = 0.22$ kips.sec/ft

(b) $\zeta = \zeta(\xi)$ $\therefore$ From P3-6c: $\zeta = 14\%$

$\blacktriangleright$ $\zeta = 14\%$

(c) Since the damping forces are nearly independent from the test frecuency, the hysteretic damping appears more reasonable.

$\blacktriangleright$ HYSTERETIC

Problem 3-8

From eq. (3-66): $E_D = \xi_{eq.}(2\pi k \rho^2)$, then if k and ρ do not change, since $\xi_{eq.} \neq \xi_{eq.}(\bar{\omega})$ then $E_D \neq E_D(\bar{\omega})$.

$\blacktriangleright$ E_D IS THE SAME

Problem 4-1 (con'd)

$$b_n = \frac{2}{T_p}\int_0^{\frac{2}{3}T_p} p_0 \sin\frac{3\pi}{T_p}t \,\sin\frac{2\pi n}{T_p}t\,dt = \frac{2p_0}{T_p}\left[\frac{\sin(3-2n)\frac{\pi t}{T_p}}{2(3-2n)\frac{\pi}{T_p}} - \frac{\sin(3+2n)\frac{\pi t}{T_p}}{2(3+2n)\frac{\pi}{T_p}}\right]_0^{\frac{2}{3}T_p}$$

$$b_n = \frac{p_0}{\pi}\cdot\frac{1}{9-4n^2}\left[(3+2n)\sin(3-2n)\frac{2}{3}\pi - (3-2n)\sin(3+2n)\frac{2}{3}\pi\right]$$

$$b_n = \frac{p_0}{\pi}\cdot\frac{1}{9-4n^2}\left[-(3+2n)\sin\frac{4\pi n}{3} - (3-2n)\sin\frac{4\pi n}{3}\right]$$

$$b_n = \frac{6p_0}{\pi}\cdot\frac{-1}{9-4n^2}\begin{cases} -\frac{\sqrt{3}}{2}, & n = 1,4,7,\dots \\ \frac{\sqrt{3}}{2}, & n = 2,5,8,\dots \\ 0, & n = 3,6,9,\dots \end{cases}$$

$$\blacktriangleright\ a_0 = 0,\quad a_n = \frac{p_0/\pi}{1-\left(\frac{2n}{3}\right)^2}\quad n \neq 3k,\quad b_n = \frac{\sqrt{3}\,p_0/(3\pi)}{1-\left(\frac{2n}{3}\right)^2}\cdot\begin{cases} 1, & n = 1,4,7,\dots \\ -1, & n = 2,5,8,\dots \end{cases}$$

$$p(t) = \frac{p_0}{\pi}\sum_{n=(1,2),(4,5),\dots}^{\infty}\frac{\cos\frac{2\pi n}{T_p}t}{1-\left(\frac{2n}{3}\right)^2} + \frac{\sqrt{3}\,p_0}{3\pi}\left[\sum_{n=1,4,\dots}^{\infty}\frac{\sin\frac{2\pi n}{T_p}t}{1-\left(\frac{2n}{3}\right)^2} - \sum_{n=2,5,\dots}^{\infty}\frac{\sin\frac{2\pi n}{T_p}t}{1-\left(\frac{2n}{3}\right)^2}\right]$$

Problem 4-2

$$p(t) = \begin{cases} p_0 & , \ 0 < t < \frac{T_p}{2} \\ -\frac{1}{2} p_0 & , \ \frac{T_p}{2} < t < T_p \end{cases}$$

From eqs. (4-1,2): $p(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{T_p} t + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n}{T_p} t$, where

$$\left. \begin{aligned} a_0 &= \frac{1}{T_p} \int_0^{T_p} p(t)dt \\ a_n &= \frac{2}{T_p} \int_0^{T_p} p(t) \cos \frac{2\pi n}{T_p} t\,dt \\ b_n &= \frac{2}{T_p} \int_0^{T_p} p(t) \sin \frac{2\pi n}{T_p} t\,dt \end{aligned} \right\} \text{Eq. (4-3)}$$

$$\therefore \ a_0 = \frac{1}{T_p} \int_0^{T_p/2} p_0\,dt + \int_{T_p/2}^{T_p} \left(-\frac{1}{2} p_0\right) dt = \frac{p_0}{T_p} \cdot \frac{T_p}{2} - \frac{p_0}{2T_p}\left(T_p - \frac{T_p}{2}\right) = \frac{p_0}{4}$$

$$a_n = \frac{2}{T_p} \int_0^{T_p/2} p_0 \cos \frac{2\pi n}{T_p} t\,dt + \frac{2}{T_p} \int_{T_p/2}^{T_p} \left(-\frac{1}{2} p_0\right) \cos \frac{2\pi n}{T_p} t\,dt$$

$$a_n = \frac{2p_0}{T_p} \left[\frac{T_p}{2\pi n} \sin \frac{2\pi n}{T_p} t\right]_0^{T_p/2} - \frac{p_0}{T_p} \left[\frac{T_p}{2\pi n} \sin \frac{2\pi n}{T_p} t\right]_{T_p/2}^{T_p} = 0$$

$$b_n = \frac{2}{T_p} \int_0^{T_p/2} p_0 \sin \frac{2\pi n}{T_p} t\,dt + \frac{2}{T_p} \int_{T_p/2}^{T_p} \left(-\frac{1}{2} p_0\right) \sin \frac{2\pi n}{T_p} t\,dt$$

$$b_n = \frac{2p_0}{T_p} \left[-\frac{T_p}{2\pi n} \cos \frac{2\pi n}{T_p} t\right]_0^{T_p/2} - \frac{p_0}{T_p} \left[-\frac{T_p}{2\pi n} \cos \frac{2\pi n}{T_p} t\right]_{T_p/2}^{T_p}$$

$$b_n = \frac{p_0}{2\pi n} \left\{ 2\left[1 - (-1)^n\right] + 1 - (-1)^n \right\}$$

(continued on following page)

Problem 4-2 (con'd)

$$b_n = \frac{3P_0}{2\pi n}\left[1+(-1)^{n+1}\right] = \begin{cases} \dfrac{3P_0}{\pi n}, & n = 1,3,\dots \\ 0, & n = 2,4,\dots \end{cases}$$

$$\blacktriangleright\ a_0 = \frac{P_0}{4},\quad a_n = 0,\quad b_n = \frac{3P_0}{\pi n}\quad n = 1,3,\dots$$

$$p(t) = \frac{P_0}{4} + \frac{3P_0}{\pi}\sum_{n=1,3,\dots}^{\infty}\frac{1}{n}\sin\frac{2\pi n}{T_p}t$$

Problem 4-3

Eq. (4-16) :
$$v(t) = \frac{1}{k}\left(a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{(1-\beta_n^2)^2 + (2\xi\beta_n)^2}\right]\cdot \left\{\left[2\xi a_n\beta_n + b_n(1-\beta_n)^2\right]\sin n\bar{\omega}_1 t + \left[a_n(1-\beta_n)^2 - 2\xi b_n\beta_n\right]\cos n\bar{\omega}_1 t\right\}\right)$$

From E4-1 : $\quad a_0 = \dfrac{P_0}{\pi}$

$$a_n = \begin{cases} 0, & n = \text{odd} \\ \dfrac{P_0}{\pi}\cdot\dfrac{2}{1-n^2}, & n = \text{even} : 2,4,\dots \end{cases}$$

$$b_n = \begin{cases} P_0/2, & n = 1 \\ 0, & n > 1 \end{cases}$$

$$\beta_1 = \frac{3}{4}$$

From eqs. (4-2, 12) : $\quad \beta_n = n\beta_1$

(continued on following page)

Problem 4-3 (con'd)

Since $\xi = 0.1$,

$$v(t) = \frac{1}{k}\left\{\frac{P_0}{\pi} + \frac{1}{\left(1-\frac{9}{16}\right)^2 + \left(2\times 0.1\times\frac{3}{4}\right)^2}\left[\frac{P_0}{2}\left(1-\frac{9}{16}\right)\sin\bar{\omega}_1 t - \frac{P_0}{2}(2)(0.1)\left(\frac{3}{4}\right)\cdot\right.\right.$$

$$\left.\cos\bar{\omega}_1 t\right] + \sum_{n=2,4,\ldots}^{\infty}\frac{1}{\left(1-\frac{9}{16}n^2\right)^2 + \left(2\times 0.1\times\frac{3}{4}n\right)^2}\cdot\frac{P_0}{\pi}\cdot\frac{2}{1-n^2}\cdot$$

$$\left.\left[2(0.1)\left(\frac{3}{4}\right)n\sin n\bar{\omega}_1 t + \left(1-\frac{9}{16}n^2\right)\cos n\bar{\omega}_1 t\right]\right\}$$

$$v(t) = \frac{P_0}{k\pi}\left\{1 + 1.02\pi\sin\bar{\omega}_1 t - 0.351\pi\cos\bar{\omega}_1 t + \right.$$

$$\sum_{n=1}^{\infty}\frac{2}{\left[\left(1-\frac{9}{4}n^2\right)^2 + (0.3n)^2\right](1-4n^2)}\left[0.3n\sin 2n\bar{\omega}_1 t + \right.$$

$$\left.\left.\left(1-\frac{9}{4}n^2\right)\cos 2n\bar{\omega}_1 t\right]\right\}$$

$$\blacktriangleright\ v(t) = \frac{P_0}{k\pi}\left\{1 + 3.21\sin\bar{\omega}_1 t - 1.10\cos\bar{\omega}_1 t + 2\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)\left[\left(1-\frac{9}{4}n^2\right)^2 + (0.3n)^2\right]}\cdot\right.$$

$$\left.\left[0.3n\sin 2n\bar{\omega}_1 t + \left(1-\frac{9}{4}n^2\right)\cos 2n\bar{\omega}_1 t\right]\right\}$$

$$v(t) = \frac{P_0}{k\pi}\left\{1 + \frac{4\pi}{3.7}\sin(\bar{\omega}_1 t - \theta_0) - \sum_{n=1}^{\infty}\frac{2\sin(2n\bar{\omega}_1 - \theta_n)}{(4n^2-1)\left[\left(\frac{9}{4}n^2-1\right)^2 + (0.3n)^2\right]^{1/2}}\right\} ;$$

$$\theta_0 = \tan^{-1}\frac{2.4}{7} , \quad \theta_n = \tan^{-1}\frac{\frac{9}{4}n^2-1}{0.3n}$$

Problem 4-4

For $p(t) = p_0 e^{i\bar{\omega}t}$, from eq. (3-28)

$$f_{I_p}(t) = -m\,\bar{\omega}^2 \rho\, e^{i(\bar{\omega}t-\theta)}$$

$$f_{D_p}(t) = i c \bar{\omega} \rho\, e^{i(\bar{\omega}t-\theta)}$$

$$f_{S_p}(t) = k \rho\, e^{i(\bar{\omega}t-\theta)}$$

where

$$\rho = \frac{p_0}{k}\left[(1-\beta^2)^2 + (2\xi\beta)^2\right]^{-1/2} \qquad \text{eq. (3-22)}$$

$$\theta = \tan^{-1}\left[\frac{2\xi\beta}{1-\beta^2}\right] \qquad \text{eq. (3-23)}$$

and

$$\beta = \frac{\bar{\omega}}{\omega}$$

$$c = 2m\omega\xi$$

$$k/m = \omega^2$$

$$\therefore \quad \rho = \left\{\left[1-\left(\frac{6}{5}\right)^2\right]^2 + \left[2(0.15)\left(\frac{6}{5}\right)\right]^2\right\}^{-1/2} \frac{p_0}{k} = 1.759\,\frac{p_0}{k}$$

$$\theta = \tan^{-1}\left[\frac{2(0.15)(6/5)}{1-(6/5)^2}\right] = \arctan\left(-\frac{9}{11}\right) = -39.3^\circ$$

$$f_{I_p}(t) = -m\,\bar{\omega}^2\, 1.759\,\frac{p_0}{k}\, e^{i(\bar{\omega}t-\theta)} = -\left(\frac{\bar{\omega}}{\omega}\right)^2 1.759\, p_0 e^{i(\bar{\omega}t-\theta)} = -2.53\, p_0 e^{i(\bar{\omega}t-\theta)}$$

$$f_{D_p}(t) = i\,2m\omega\xi\bar{\omega}\, 1.759\,\frac{p_0}{k}\, e^{i(\bar{\omega}t-\theta)} = i\,2\frac{\bar{\omega}}{\omega}\xi\, 1.759\, p_0 e^{i(\bar{\omega}t-\theta)} = i\,0.633\, p_0 e^{i(\bar{\omega}t-\theta)}$$

$$f_{S_p}(t) = k\, 1.759\,\frac{p_0}{k}\, e^{i(\bar{\omega}t-\theta)} = 1.759\, p_0 e^{i(\bar{\omega}t-\theta)}$$

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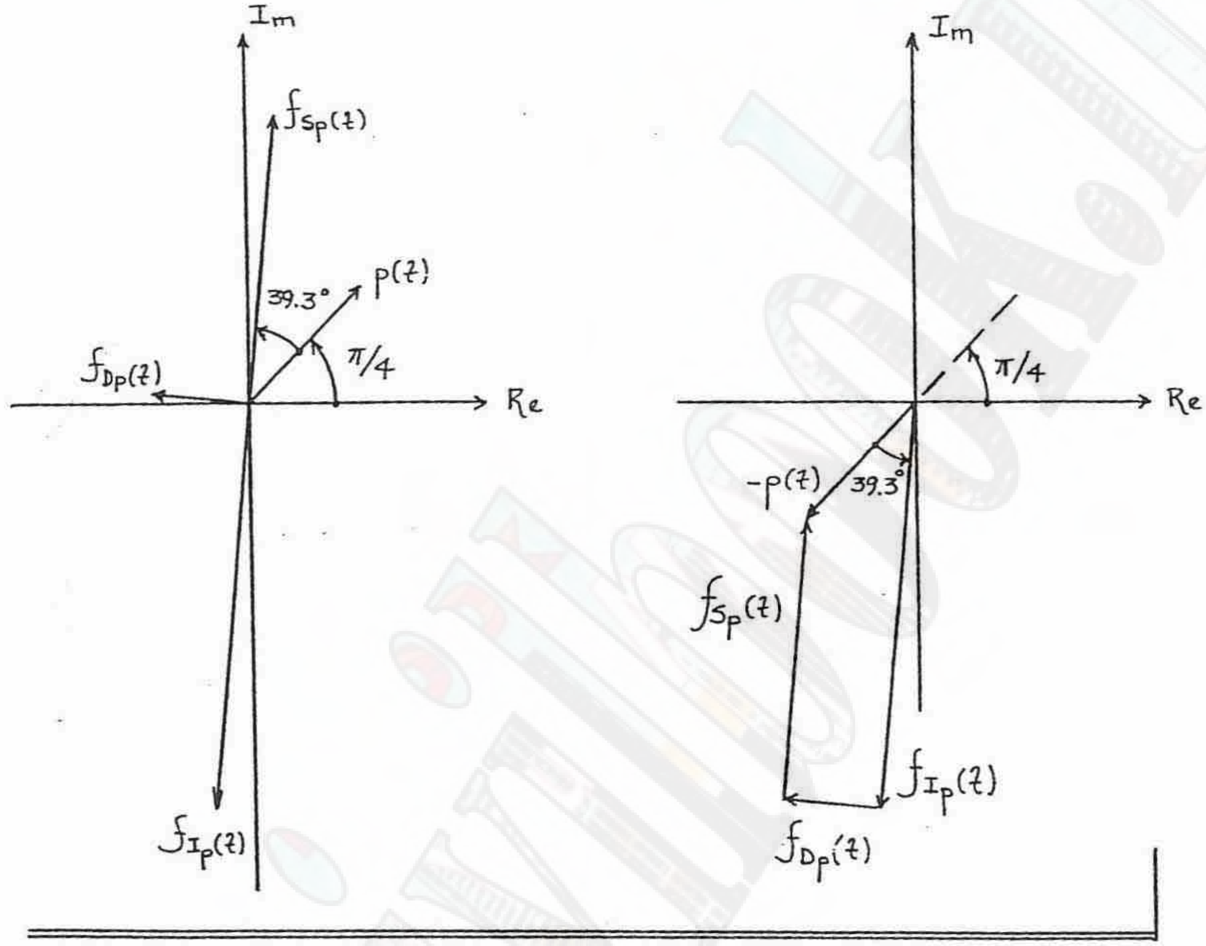
Problem 4-4 (con'd)



Problem 4-5

For $p(t) = \sum_{n=1}^{\infty} b_n \sin \overline{\omega}_n t \quad , \quad b_n = -\dfrac{2\,p_o}{n\pi}(-1)^n$

From eq. (3-21): $v(t) = \sum_{n=1}^{\infty} \rho_n \sin(\overline{\omega}_n t - \theta_n)$, where

$$\rho_n = \frac{b_n}{k}\left[\left(1-\beta_n^2\right)^2 + \left(2\,\xi\beta_n\right)^2\right]^{-1/2} \qquad \text{eq. (3-22)}$$

$$\theta_n = \tan^{-1}\left[\frac{2\,\xi\,\beta_n}{1-\beta_n^2}\right] \qquad \text{eq. (3-23)}$$

(continued on following page)

Problem 5-1

It will be assumed that $c = 0$.

(a) First eq. of E5-1 : $T = 2\pi\sqrt{\frac{W}{kg}} = 2\pi\sqrt{\frac{600}{1000(386)}} = 0.248 \text{ sec}$

Since $\beta = \frac{\bar{\omega}}{\omega} = \frac{T}{\bar{T}} = \frac{T}{2t_1} = \frac{0.248}{2(0.15)} = 0.826 < 1 \longrightarrow v_{max}$ is in phase I

From eq. (5-8) $\alpha_{R_{max}} = \frac{t_o}{t_1} = \frac{\beta}{\beta+1} = \frac{2}{1+(1/\beta)}$

$$\therefore\ t_o = \frac{2t_1}{1+\frac{2t_1}{T}} = \frac{2(0.15)}{1+\frac{2(0.15)}{0.248}} = 0.136 \text{ sec}$$

$\blacktriangleright\ t_o = 0.14 \text{ sec}$

(b) Last eq. of E5-1 : $f_{S,max} = k\, v_{max}$

– From eq. (5-1) : $v_{max} = \frac{P_o}{k} R(\alpha_{R_{max}})$

$$v_{max} = \frac{P_o}{k}\cdot\frac{1}{1-\beta^2}\left(\sin\pi\alpha - \beta\sin\frac{\pi\alpha}{\beta}\right),\ \alpha = \frac{2}{1+(1/\beta)}$$

$$\therefore f_{S,max} = \frac{P_o}{1-\beta^2}\left(\sin\frac{2\pi}{1+(1/\beta)} - \beta\sin\frac{2\pi}{1+\beta}\right)$$

$$f_{S,max} = \frac{500^{lb}}{1-(0.826)^2}\left(\sin\frac{2\pi}{1+(1/0.826)} - 0.826\sin\frac{2\pi}{1+0.826}\right) = 847^{lb}$$

– or from third eq. of E5-1 : $v_{max} = R_{max}\left(\frac{P_o}{k}\right)$

$$\therefore\ \hat{f}_{S,max} = p_o R_{max}$$

$\frac{t_1}{T} = \frac{0.15}{0.248} = 0.605 \xrightarrow{\text{Fig. 5-6}} R_{max} = 1.68$

$$\hat{f}_{S,max} = 500^{lb}(1.68) = 840^{lb}$$

$\blacktriangleright\ f_{S,max} = 850^{lb}\ ,\ \hat{f}_{S,max} = 840^{lb}$

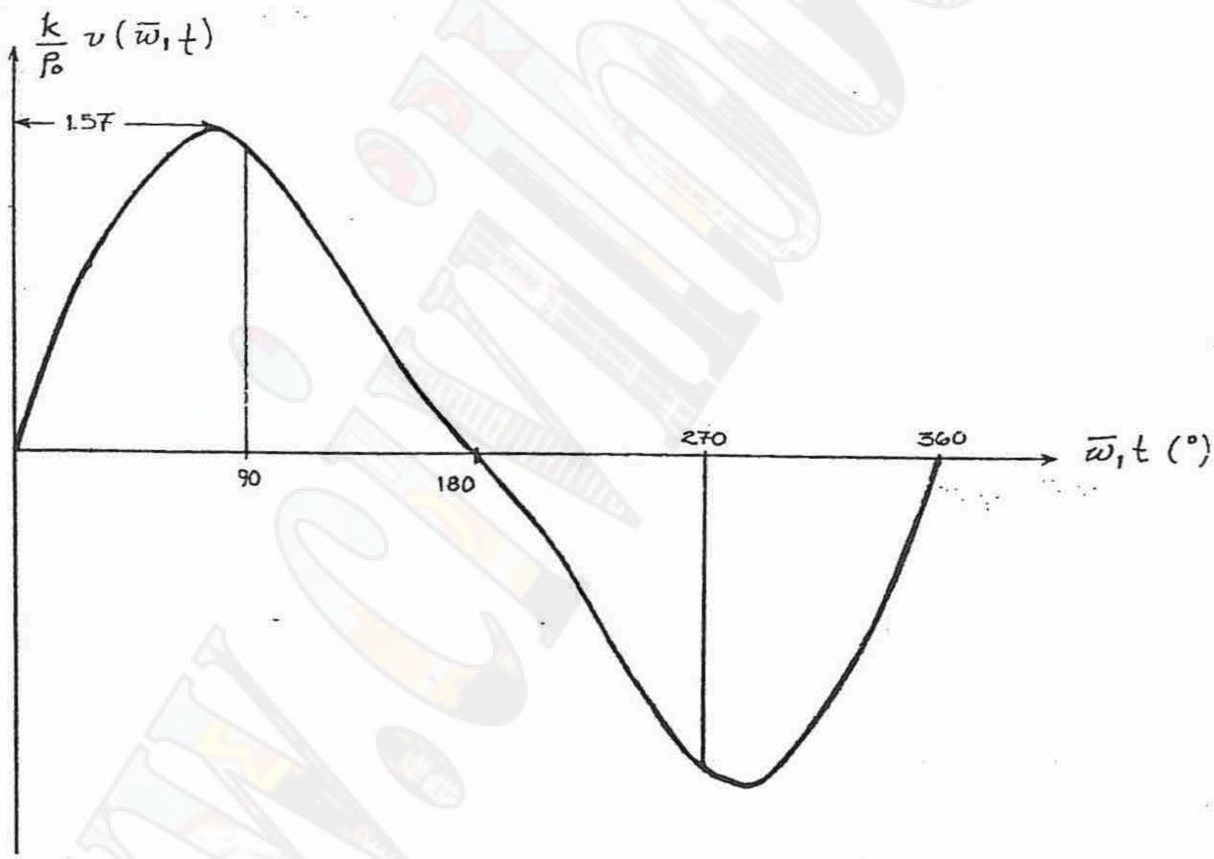
Problem 4-5 (con'd)

Assuming the same properties of E4-1, i.e., $\xi = 0$, $\beta_1 = \frac{3}{4}$

$$v(\bar{\omega}_1 t) = \frac{2P_0}{\pi k} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\left(1-\frac{9}{16}n^2\right)} \sin(n\bar{\omega}_1 t)$$

$$v(\bar{\omega}_1 t) = \frac{2P_0}{\pi k}\left(\frac{16}{7}\sin\bar{\omega}_1 t + \frac{2}{5}\sin 2\bar{\omega}_1 t - \frac{16}{195}\sin 3\bar{\omega}_1 t + \frac{1}{32}\sin 4\bar{\omega}_1 t \mp \cdots\right)$$

$$v(\bar{\omega}_1 t) = \frac{P_0}{k}\left(1.455\sin\bar{\omega}_1 t + 0.255\sin 2\bar{\omega}_1 t - 0.052\sin 3\bar{\omega}_1 t + 0.020\sin 4\bar{\omega}_1 t \mp \cdots\right)$$



$\bar{\omega}_1 t$ (°)	0	30	60	90	120	150	180	210	240	270	300	330	360
$\frac{k}{P_0} v(\bar{\omega}_1 t)$	0	.914	1.464	1.507	1.057	.437	0	-.437	-1.057	-1.507	-1.464	-.914	0

Problem 5-3 (con'd)

Then if v_{max} is in phase I, $\dot{v}(t) = \frac{p_0 \omega}{2k}\left(\sin \omega t + \omega t \cos \omega t\right) = 0$

$$\therefore \sin \omega t + \omega t \cos \omega t = 0$$

$$\tan \omega t = -\omega t$$

$$\therefore \omega t = -\omega t + n\pi \ , \ n = 0, \pm 1, \pm 2, \ldots$$

But $0 < t < t_1 \longrightarrow 0 < \omega t < \omega t_1 = \frac{\pi}{2} \longrightarrow n = 1$

$$\therefore \omega t_0 = \frac{\pi}{2} \ \rightarrow\!\!\!\times\!\!\!\leftarrow$$

The maximum is in phase II, $\therefore R_{max} = \frac{v_{max}}{p_0/k} = \frac{1}{2}\sqrt{1 + \frac{\pi^2}{4}}$

$$\blacktriangleright R_{max} = \frac{1}{2}\sqrt{1 + \frac{\pi^2}{4}}$$

Problem 5-4

(a) From the two last eqs. of E5-1: $f_{S,max} = k\, v_{max} = R_{max}\, p_0$

From Fig. 5-6, $t_1/T = 0.15 \longrightarrow R_{max} = 0.46$

$$\therefore f_{S,max}^{(a)} = 0.46(15) = 6.9 \text{ kips}$$

$$\blacktriangleright f_{S,max}^{(a)} = 6.9 \text{ kips}$$

(b) Eq. (5-21): $v(\bar{t}) \doteq \frac{1}{m\omega}\left[\int_0^{t_1} p(t)\,dt\right] \sin \omega \bar{t}$

$$\therefore v_{max} = \frac{1}{m\omega}\int_0^{t_1} p(t)\,dt = \frac{1}{m\omega}\int_0^{t_1} p_0\left(1 - \frac{t}{t_1}\right)dt$$

$$v_{max} = \frac{p_0}{m\omega}\left(t_1 - \frac{1}{2}t_1\right) = \frac{p_0 t_1}{2m\omega}$$

(continued on following page)

Problem 5-4 (con'd)

Including $T = 2\pi/\omega$ and $\omega^2 = k/m$ in v_{max}

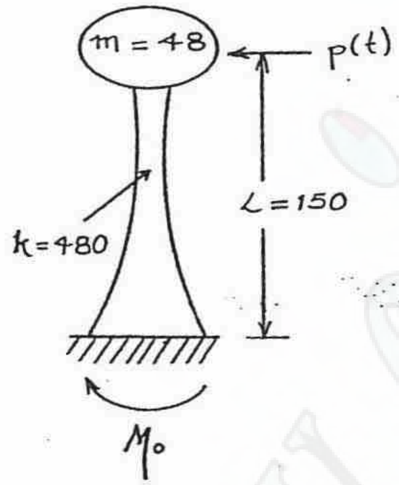
$$v_{max} = \pi \frac{t_1}{T} \frac{p_0}{k} = \pi (0.15) \frac{15}{20} = 0.353 \text{ in}$$

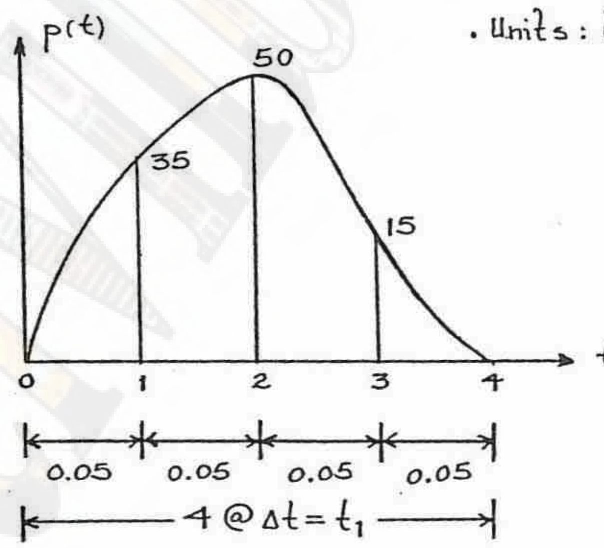
$$\therefore f_{s,max}^{(b)} = \pi \frac{t_1}{T} p_0 = \pi (0.15)(15) = 7.07 \text{ kips}$$

Then $$\frac{f_{s,max}^{(a)}}{f_{s,max}^{(b)}} = \frac{R_{max}}{\pi} \frac{T}{t_1} = \frac{0.46}{\pi} \frac{1}{0.15} = 0.98$$

▶ $v_{max} = 0.35\text{in}$, $f_{s,max}^{(b)} = 7.1 \text{ kips}$, $\dfrac{f_{s,max}^{(a)}}{f_{s,max}^{(b)}} = 0.98$

Problem 5-5





. Units: kips-ft

$$M_0 = f_{s,max} L = (k\, v_{max}) L$$

$$\frac{t_1}{T} = \frac{t_1}{2\pi\sqrt{m/k}} = 0.10 < 1/4$$, thus p(t) is a short duration load

$$\therefore v(t) \doteq \frac{1}{m\omega}\left[\int_0^{t_1} p(\bar{t})d\bar{t}\right] \sin\omega\bar{t} \qquad \text{eq. (5-21)}$$

$$v_{max} = \frac{1}{m\omega}\int_0^{t_1} p(\bar{t})d\bar{t} \quad , \quad M_0 = \frac{kL}{m\sqrt{k/m}}\int_0^{t_1} p(\bar{t})d\bar{t}$$

▶ $$M_0 \approx L\sqrt{k/m}\,\frac{\Delta t}{3}\left(p_0 + 4p_1 + 2p_2 + 4p_3 + p_4\right) = \underline{2370 \text{ kips}\cdot\text{ft}}$$

Problem 5-2

The system will be assumed undamped.

(a) A SDOF has to verify: $\quad m\dot{v}(t) + k\,v(t) = p(t)\ , \quad c = 0$

$$p(t) = \begin{cases} p_0 \dfrac{t}{t_1}\ , & 0 < t < t_1 \\[2ex] 0\ , & \bar{t} = t - t_1 \geq 0 \end{cases}$$

- Phase I: $0 < t < t_1$,

Complementary solution: $v_c(t) = A\sin\omega t + B\cos\omega t$

Particular solution: $v_p(t) = \dfrac{p_0}{k}\cdot\dfrac{t}{t_1}$

General solution: $v(t) = A\sin\omega t + B\cos\omega t + \dfrac{p_0}{k}\dfrac{t}{t_1}$

$\therefore\quad \dot{v}(t) = A\omega\cos\omega t - B\omega\sin\omega t + \dfrac{p_0}{k t_1}$

Starting from "at rest",
$$\left.\begin{aligned} v(0) &= 0 = B \\ \dot{v}(0) &= 0 = A\omega + \frac{p_0}{k t_1} \end{aligned}\right\} \longrightarrow A = -\frac{p_0}{k\omega t_1}\ ,\ B = 0$$

Therefore, $\quad v(t) = \dfrac{p_0}{k}\left(\dfrac{t}{t_1} - \dfrac{1}{\omega t_1}\sin\omega t\right)$

- Phase II: $t \geq t_1$,

$$v(\bar{t}) = \frac{\dot{v}(t_1)}{\omega}\sin\omega\bar{t} + v(t_1)\cos\omega\bar{t}\ , \quad \bar{t} = t - t_1 \geq 0;$$

see eq. (2-33)

where
$$v(t_1) = \frac{p_0}{k}\left(1 - \frac{1}{\omega t_1}\sin\omega t_1\right)$$
$$\dot{v}(t_1) = \frac{p_0}{k t_1}\left(1 - \cos\omega t_1\right) = \frac{2p_0}{k t_1}\sin^2\frac{\omega t_1}{2}$$

$$\blacktriangleright\ v(t) = \begin{cases} \dfrac{p_0}{k}\left(\dfrac{t}{t_1} - \dfrac{1}{\omega t_1}\sin\omega t\right)\ , & 0 < t < t_1 \\[2ex] \dfrac{p_0}{k}\left[\dfrac{2}{\omega t_1}\sin^2\dfrac{\omega t_1}{2}\sin\omega(t - t_1) + \left(1 - \dfrac{1}{\omega t_1}\sin\omega t_1\right)\cos\omega(t - t_1)\right]\ , & t \geq t_1 \end{cases}$$

(continued on following page)

Problem 5-2 (con'd)

(b) If v_{max} is in phase I: $\dot{v}(t) = \frac{p_0}{k t_1}\left(1 - \cos\omega t\right)$

$$\dot{v}(t) = \frac{2 p_0}{k t_1} \sin^2 \frac{\omega t_1}{2} = 0, \quad 0<t<t_1$$

$$\therefore \sin \frac{\omega t}{2} = 0 \longrightarrow \frac{\omega t}{2} = \eta\pi, \quad \eta = 0, \pm 1, \pm 2, \ldots$$

since $t > 0$, $\eta = 1, 2, \ldots$

Let us chose the first maximum $\longrightarrow t = \frac{2\pi}{\omega} < t_1$

Then, the maximum will be in phase I if $t_1 > \frac{2\pi}{\omega}$.

In this case, $t_1 = \frac{3\pi}{\omega}$, then the maximum is in phase I, when $t = \frac{2\pi}{\omega} = \frac{2}{3} t_1$

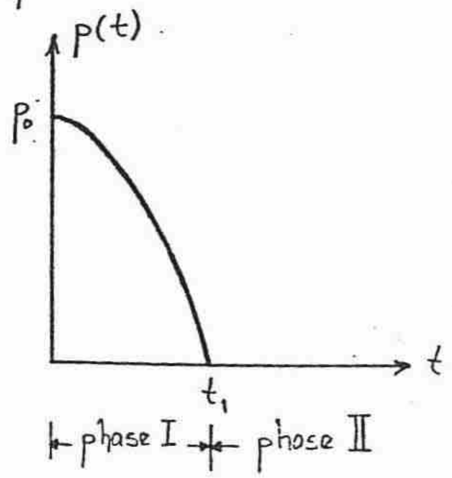
$$\therefore R_{max} = \frac{v_{max}}{p_0/k} = \frac{2}{3} - \frac{1}{3\pi} \sin\left(\frac{3\pi}{t_1} - \frac{2}{3} t_1\right) = \frac{2}{3}$$

$$\blacktriangleright \boxed{R_{max} = \frac{2}{3}}$$

Problem 5-3

The SDOF system will be assumed undamped.

$$p(t) = \begin{cases} p_0 \cos \bar{\omega} t, & 0 < t \le t_1 = \frac{\pi}{2\bar{\omega}} \\ 0, & t_1 \le t \end{cases}$$



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Problem 5-3 (con'd)

(a) Phase I: $p(t) = p_o \cos\bar{\omega}t + p_o \sin\left(\bar{\omega}t + \frac{\pi}{2}\right)$

From eq. (3-8): $v(t) = A\sin\omega t + B\cos\omega t + \frac{p_o}{k}\left[\frac{1}{1-\beta^2}\right]\sin\left(\bar{\omega}t + \frac{\pi}{2}\right)$

$$v(t) = A\sin\omega t + B\cos\omega t + \frac{p_o}{k}\left[\frac{1}{1-\beta^2}\right]\cos\bar{\omega}t,$$

where $\beta \equiv \bar{\omega}/\omega$ eq. (3-7)

$$\therefore \dot{v}(t) = A\omega\cos\omega t - B\omega\sin\omega t - \frac{p_o}{k}\frac{\bar{\omega}}{1-\beta^2}\sin\bar{\omega}t$$

Starting from "at rest":

$$\left.\begin{aligned} v(0) &= 0 = B + \frac{p_o}{k}\cdot\frac{1}{1-\beta^2} \\ \dot{v}(0) &= 0 = A\omega \end{aligned}\right\} \longrightarrow A = 0,\; B = \frac{p_o/k}{1-\beta^2}$$

Therefore, $v(t) = \frac{p_o}{k}\left[\frac{1}{1-\beta^2}\right]\left(\cos\bar{\omega}t - \cos\omega t\right)$

Phase II: $t \geq t_1$

$v(\bar{t}) = \frac{\dot{v}(t_1)}{\omega}\sin\omega\bar{t} + v(t_1)\cos\omega\bar{t}$, $\bar{t} = t - t_1 \geq 0$ from eq. (2-33)

where $v(t_1) = \frac{p_o}{k}\cdot\frac{-1}{1-\beta^2}\cos\frac{\pi}{2\beta}$

$$\dot{v}(t_1) = \frac{p_o}{k}\frac{\omega}{1-\beta^2}\left(\sin\frac{\pi}{2\beta} - \beta\right)$$

$$\blacktriangleright\; v(t) = \begin{cases} \frac{p_o/k}{1-\beta^2}\left(\cos\beta\omega t - \cos\omega t\right), & 0 < t < t_1 = \frac{\pi}{2\beta\omega} \\ \frac{p_o/k}{1-\beta^2}\left[\left(\sin\frac{\pi}{2\beta} - \beta\right)\sin\omega(t-t_1) - \cos\frac{\pi}{2\beta}\cos\omega(t-t_1)\right], & t_1 \leq t \end{cases}$$

(continued on following page)

Problem 5-3 (con'd)

(b) If v_{max} is in phase I : $\dot{v}(t) = \frac{P_o}{k}\frac{w}{1-\beta^2}\left(\sin wt - \beta \sin \bar{w}t\right) = 0$

$$\therefore \quad \sin wt = \beta \sin \bar{w} t$$

However, when $w = \bar{w}$ ($\beta = 1$) the above expressions are not valid anymore. Then, applying L'Hôpital rule at the expressions of part (a)

$$\lim_{\beta\to 1} v(t) = \lim_{\beta\to 1} \frac{P_o/k}{1-\beta^2}\left(\cos\beta wt - \cos wt\right) = \lim_{\beta\to 1}\frac{P_o/k\,(-wt\sin\beta wt)}{-2\beta}$$

$$\lim_{\beta\to 1} v(t) = \lim_{\beta\to 1} \frac{P_o wt}{2\beta k}\sin\beta wt$$

$$\therefore\ v(t) = \frac{P_o wt}{2k}\sin wt \longrightarrow v(t_1) = \frac{P_o w t_1}{2k}\sin wt_1 = \frac{P_o\pi}{4k}$$

$$\dot{v}(t) = \frac{P_o w}{2k}\left(\sin wt + wt\cos wt\right) \longrightarrow \dot{v}(t_1) = \frac{P_o w}{2k}\left(\sin wt_1 - wt_1\cos wt_1\right)$$

$$\dot{v}(t_1) = \frac{P_o w}{2k}$$

Therefore, in phase II, $v(\bar{t})$ will be :

$$v(\bar{t}) = \frac{P_o}{2k}\sin w\bar{t} + \frac{P_o\pi}{4k}\cos w\bar{t} \quad , \quad \bar{t} = t - t_1 \geq 0$$

$$v(\bar{t}) = \frac{P_o}{2k}\sqrt{1+\frac{\pi^2}{4}}\ \sin(w\bar{t} - \theta) \, , \quad \theta = \tan^{-1}\left(-\frac{\pi}{2}\right) = 57.5°$$

$$\blacktriangleright\ \boxed{v(t) = \begin{cases} \dfrac{P_o wt}{2k}\sin wt & , \ 0 < t < t_1 \\ \dfrac{P_o}{2k}\sqrt{1+\dfrac{\pi^2}{4}}\ \sin\left[w(t-t_1) + \tan^{-1}\dfrac{\pi}{2}\right] & , \ t \geq t_1 \end{cases}}$$

(continued on following page)

Note to chapter 6:

The solutions to the chapter 6 problems were carried out using a computer program available through F. Medina. The results presented on this Solution Manual are computed in the same fashion as those obtained for Examples E6-1 and E-2, respectively, but applied to the corresponding method (simple summation, trapezoidal rule, and Simpson's rule). However, the computations were carried out using the damped natural frequency ω_D, which modifies the following:

col. (2): $\sin \omega_D t_N$

col. (3): $\cos \omega_D t_N$

$$F = \frac{\Delta\tau}{\zeta m \omega_D}, \quad \zeta = \begin{cases} 1, & \text{simple summation} \\ 2, & \text{trapezoidal rule} \\ 3, & \text{Simpson's rule} \end{cases}$$

This difference would only affects the results to problems P6-3 and P6-4, slightly.

Problem 6-1

$$f_s(t) = k\,v(t)$$

From eq. (5-1): $f_s(t) = k\,\frac{P_0}{k}\cdot\frac{1}{1-\beta^2}\left(\sin\pi\alpha - \beta\sin\frac{\pi\alpha}{\beta}\right)$

where $\alpha = t/t_1 \longrightarrow$

$$\beta = \frac{T}{2t_1} = \frac{\pi}{\omega t_1} = \frac{\pi}{\sqrt{\frac{k}{m}}\,t_1} = \frac{\pi}{\sqrt{\frac{\pi^2\ \mathrm{lb/in}}{\frac{1}{4}\ \mathrm{lb\cdot sec^2/in}}}\,t_1} = \frac{1}{\frac{2}{\mathrm{sec}}\,t_1}$$

$$\therefore f_s(t) = \frac{P_0}{1-\left(\frac{1}{2t_1}\right)^2}\left(\sin\pi\frac{t}{t_1} - \frac{1}{2t_1}\sin 2\pi t\right) \qquad t_1 \text{ in sec.}$$

$$f_s(t) = 327\left(\sin\frac{\pi}{0.6}t - \frac{1}{1.2}\sin 2\pi t\right)$$

(a) Simple summation:

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.22
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
1	0.100	50.00	0.5878	0.8090	40.45	0.00	0.00	0.00	29.39	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
2	0.200	86.60	0.9511	0.3090	26.76	40.45	0.00	40.45	82.36	29.39	0.00	29.39	38.47	9.08	29.39	1.871	18.466
3	0.300	100.00	0.9511	-0.3090	-30.90	26.76	40.45	67.21	95.11	82.36	29.39	111.75	63.92	-34.53	98.46	6.268	61.862
4	0.400	86.60	0.5878	-0.8090	-70.06	-30.90	67.21	36.31	50.90	95.11	111.75	206.86	21.34	-167.35	188.70	12.013	118.561
5	0.500	50.00	0.0000	-1.0000	-50.00	-70.06	36.31	-33.75	0.00	50.90	206.86	257.76	0.00	-257.76	257.76	16.410	161.957
6	0.600	0.00	-0.5878	-0.8090	0.00	-50.00	-33.75	-83.75	0.00	0.00	257.76	257.76	49.23	-208.53	257.76	16.410	161.957

(b) Trapezoidal rule:

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.22
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
1	0.100	50.00	0.5878	0.8090	40.45	0.00	0.00	40.45	29.39	0.00	0.00	29.39	23.78	23.78	0.00	0.000	0.000
2	0.200	86.60	0.9511	0.3090	26.76	40.45	40.45	107.66	82.36	29.39	29.39	141.14	102.39	43.62	58.78	1.871	18.466
3	0.300	100.00	0.9511	-0.3090	-30.90	26.76	107.66	103.52	95.11	82.36	141.14	318.61	98.46	-98.46	196.91	6.268	61.862
4	0.400	86.60	0.5878	-0.8090	-70.06	-30.90	103.52	2.56	50.90	95.11	318.61	464.62	1.50	-375.89	377.39	12.013	118.561
5	0.500	50.00	0.0000	-1.0000	-50.00	-70.06	2.56	-117.50	0.00	50.90	464.62	515.53	0.00	-515.53	515.53	16.410	161.957
6	0.600	0.00	-0.5878	-0.8090	0.00	-50.00	-117.50	-167.50	0.00	0.00	515.53	515.53	98.46	-417.07	515.53	16.410	161.957

(continued on following page)

Problem 6-1 (con'd)

(c) Simpson's rule:

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.22
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
1	0.100	50.00	0.5878	0.8090	40.45				29.39								
2	0.200	86.60	0.9511	0.3090	26.76	161.80	0.00	188.57	82.36	117.56	0.00	199.92	179.34	61.78	117.56	2.495	24.621
3	0.300	100.00	0.9511	-0.3090	-30.90				95.11								
4	0.400	86.60	0.5878	-0.8090	-70.06	-123.61	215.33	21.66	50.90	380.42	282.28	713.61	12.73	-577.32	590.05	12.521	123.580
5	0.500	50.00	0.0000	-1.0000	-50.00				0.00								
6	0.600	0.00	-0.5878	-0.8090	0.00	-200.00	-48.41	-248.41	0.00	0.00	764.51	764.51	146.01	-618.51	764.51	16.224	160.120

Method	t, sec	0.0	0.1	0.2	0.3	0.4	0.5	0.6
a. Simple Summation	f_s, lb	0	0	18.5	61.9	119	162	162
b. Trapezoidal Rule		0	0	18.5	61.9	119	162	162
c. Simpson's Rule		0	—	24.6	—	124	—	160
eq. (5-1)		0	3.3	24.0	67.9	123	164	160

Problem 6-2

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.2
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.00
1	0.005	19.32	0.1494	0.9888	19.10	0.00	0.00	19.10	2.89	0.00	0.00	2.89	2.85	2.85	0.00	0.000	0.00
2	0.010	38.64	0.2955	0.9553	36.91	19.10	19.10	75.12	11.42	2.89	2.89	17.19	22.20	16.43	5.77	0.160	0.43
3	0.015	57.96	0.4350	0.9004	52.19	36.91	75.12	164.22	25.21	11.42	17.19	53.82	71.43	48.46	22.97	0.638	1.72
4	0.020	77.28	0.5646	0.8253	63.78	52.19	164.22	280.20	43.64	25.21	53.82	122.67	158.21	101.24	56.97	1.582	4.273
5	0.025	96.60	0.6816	0.7317	70.68	63.78	280.20	414.66	65.85	43.64	122.67	232.15	282.65	169.86	112.79	3.133	8.459
6	0.030	77.28	0.7833	0.6216	48.04	70.68	414.66	533.38	60.54	65.85	232.15	358.53	417.81	222.87	194.94	5.415	14.621
7	0.035	57.96	0.8674	0.4976	28.84	48.04	533.38	610.26	50.28	60.54	358.53	469.34	529.35	233.53	295.82	8.217	22.186
8	0.040	38.64	0.9320	0.3624	14.00	28.84	610.26	653.10	36.01	50.28	469.34	555.63	608.71	201.34	407.37	11.316	30.553
9	0.045	19.32	0.9757	0.2190	4.23	14.00	653.10	671.33	18.85	36.01	555.63	610.50	655.03	133.70	521.33	14.481	39.100
10	0.050	0.00	0.9975	0.0707	0.00	4.23	671.33	675.56	0.00	18.85	610.50	629.35	673.87	44.52	629.35	17.482	47.201

Third eq. of E6-1: $v_{max} = \left[(\overline{A}^*)^2 + (\overline{B}^*)^2 \right]^{1/2}$

From table above: $v_{max} = \left[(675.56)^2 + (629.35)^2 \right]^{1/2} \cdot F$

$$F = \frac{\Delta\tau}{2m\omega} = \frac{\Delta\tau}{2\sqrt{k\,(W/g)}} = \frac{0.005}{2\sqrt{2700\,(96.6/32.2)}}$$

$\therefore\ v_{max} = 0.025647\ ft$

$\therefore\ f_{s,max} = 2700\,(0.025647) = 69.247^{\text{kips}}$

$\blacktriangleright\ \boxed{f_{s,max} = 69.2^{\text{kips}}}$

Problem 6-3

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.22
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
1	0.005	19.32	0.1493	0.9888	19.10	0.00	0.00	19.10	2.88	0.00	0.00	2.88	2.85	2.85	0.00	0.000	0.000
2	0.010	38.64	0.2952	0.9554	36.92	18.96	18.96	74.84	11.41	2.86	2.86	17.13	22.09	16.37	5.72	0.159	0.430
3	0.015	57.96	0.4345	0.9007	52.20	36.64	74.28	163.13	25.18	11.32	17.00	53.50	70.87	48.19	22.68	0.631	1.703
4	0.020	77.28	0.5640	0.8258	63.81	51.81	161.91	277.54	43.59	24.99	53.10	121.68	156.54	100.48	56.06	1.559	4.210
5	0.025	96.60	0.6810	0.7323	70.74	63.34	275.46	409.54	65.78	43.26	120.77	229.82	278.88	168.30	110.58	3.076	8.304
6	0.030	77.28	0.7826	0.6225	48.11	70.21	406.48	524.80	60.48	65.29	228.10	353.87	410.73	220.28	190.45	5.297	14.301
7	0.035	57.96	0.8668	0.4987	28.91	47.75	520.88	597.54	50.24	60.03	351.22	461.49	517.93	230.15	287.77	8.004	21.610
8	0.040	38.64	0.9315	0.3638	14.06	28.69	593.07	635.82	35.99	49.86	458.04	543.90	592.26	197.85	394.41	10.970	29.618
9	0.045	19.32	0.9754	0.2207	4.26	13.95	631.06	649.28	18.84	35.72	539.84	594.40	633.27	131.16	502.12	13.965	37.706
10	0.050	0.00	0.9974	0.0726	0.00	4.23	644.43	648.66	0.00	18.70	589.96	608.66	646.95	44.19	602.75	16.764	45.263
11	0.055	0.00	0.9970	-0.0771	0.00	0.00	643.81	643.81	0.00	0.00	604.12	604.12	641.90	-46.56	688.45	19.148	51.699
12	0.060	0.00	0.9744	-0.2250	0.00	0.00	639.00	639.00	0.00	0.00	599.60	599.60	622.61	-134.92	757.53	21.069	56.886
13	0.065	0.00	0.9299	-0.3679	0.00	0.00	634.23	634.23	0.00	0.00	595.12	595.12	589.74	-218.95	808.70	22.492	60.728
14	0.070	0.00	0.8645	-0.5026	0.00	0.00	629.49	629.49	0.00	0.00	590.68	590.68	544.21	-296.86	841.07	23.392	63.159
15	0.075	0.00	0.7798	-0.6260	0.00	0.00	624.78	624.78	0.00	0.00	586.26	586.26	487.23	-366.99	854.22	23.758	64.147
16	0.080	0.00	0.6777	-0.7354	0.00	0.00	620.12	620.12	0.00	0.00	581.88	581.88	420.24	-427.89	848.13	23.589	63.689
17	0.085	0.00	0.5603	-0.8283	0.00	0.00	615.48	615.48	0.00	0.00	577.53	577.53	344.87	-478.35	823.23	22.896	61.819
18	0.090	0.00	0.4304	-0.9026	0.00	0.00	610.88	610.88	0.00	0.00	573.22	573.22	262.94	-517.40	780.34	21.703	58.599

The maximum response is reached very close to $t = 0.075$ sec with $v_{max} \doteq 0.023758$ and $f_{s,max} = 64.147$ kips

$\blacktriangleright\ \boxed{f_{s,max} = 64.1\ \text{kips}}$

Problem 6-4

N	T(N)	COL. 1	COL. 2	COL. 3	COL. 4	COL. 7	COL. 8	COL.10	COL.11	COL.14	COL.15	COL.17	COL.18	COL.19	COL.20	COL.21	COL.22
0	0.000	0.00	0.0000	1.0000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000	0.000
1	0.120	1.00	0.6812	0.7321	0.73				0.68								
2	0.240	4.00	0.9974	0.0719	0.29	2.60	0.00	2.88	3.99	2.42	0.00	6.41	2.88	0.46	2.42	0.077	0.619
3	0.360	9.00	0.7792	-0.6268	-5.64				7.01								
4	0.480	9.00	0.1435	-0.9897	-8.91	-20.01	2.50	-26.42	1.29	24.88	8.18	34.35	-3.79	-33.99	30.20	0.967	7.738
5	0.600	6.00	-0.5691	-0.8223	-4.93				-3.41								
6	0.720	0.00	-0.9768	-0.2143	0.00	-17.50	-27.79	-45.29	0.00	-12.11	28.04	15.92	44.24	-3.41	47.65	1.526	12.209

t, sec	0	0.24	0.48	0.72
v, in	0	0.077	0.970	1.530

Note to chapter 7 :

The solutions to the chapter 7 problems were carried out using a computer program available through F. Medina. The results presented on this Solution Manual are computed in the same fashion as those obtained for the Example E7-2, shown on Table E7-1 in the book, but for columns (16) to (19), which express:

col. (16) : $\Delta \ddot{v} = \frac{6}{h^2} \Delta v - \frac{6}{h} \dot{v}_o - 3 \ddot{v}_o$, from eq. (b) of Fig. 7-4

col. (17): $\Delta \dot{v} = \frac{3}{h} \Delta v - 3 \dot{v}_o - \frac{h}{2} \ddot{v}_o$, eq. (7-24c)

$$\Delta \dot{v} = \frac{h}{2} \Delta \ddot{v} + h \ddot{v}_o$$

col. (18) and col. (19): eliminated.

Problem 7-1

Following E7-2:

$$\tilde{k}(t) = k(t) + \frac{6}{(\Delta t)^2} m + \frac{3}{\Delta t} c$$

$$\tilde{k}(t) = 8 + \frac{6}{(0.12)^2}(0.2) + \frac{3}{0.12}(0.4) = \frac{304}{3} \text{ kips/in}$$

$$\Delta \tilde{p}(t) = \Delta p(t) + \left(\frac{6m}{\Delta t} + 3c\right)\dot{v}_0 + \left(3m + \frac{\Delta t}{2} c\right)\ddot{v}_0$$

$$\Delta \tilde{p}(t) = \Delta p(t) + \left[\frac{6}{0.12}(0.2) + 3(0.4)\right]\dot{v}_0 + \left[3(0.2) + \frac{0.12}{2}(0.4)\right]\ddot{v}_0$$

$$\Delta \tilde{p}(t) = \Delta p(t) + 11.2\,\dot{v}_0 + 0.624\,\ddot{v}_0$$

$$\Delta \dot{v} = \frac{3}{\Delta t}\Delta v - 3\dot{v}_0 - \frac{\Delta t}{2}\ddot{v}_0 = \frac{3}{0.12}\Delta v - 3\dot{v}_0 - \frac{0.12}{6}\ddot{v}_0$$

$$\Delta \dot{v} = 25\,\Delta v - 3\dot{v}_0 - 0.06\,\ddot{v}_0$$

COL. 1	COL. 2	COL. 3	COL. 4	COL. 5	COL. 6	COL. 7	COL. 8	COL. 9	COL.10	COL.11	COL.12	COL.13	COL.14	COL.15	COL.16	COL.17
0.000	0.000	0.000	0.0000	0.0000	0.0000	0.0000	0.0000	1.000	0.0000	0.00	1.000	8.0	101.3	0.010	4.1118	0.2467
0.120	1.000	0.010	0.2467	0.0789	0.0987	0.8224	4.1118	3.000	0.3947	4.93	8.329	8.0	101.3	0.082	9.5763	1.0680
0.240	4.000	0.092	1.3147	0.7365	0.5259	2.7376	13.6881	5.000	1.9062	21.36	28.266	8.0	101.3	0.279	9.4261	2.2081
0.360	9.000	0.371	3.5228	2.9680	1.4091	4.6228	23.1142	0.000	4.7822	49.10	53.879	8.0	101.3	0.532	-23.9424	1.3372
0.480	9.000	0.903	4.8600	7.2216	1.9440	-0.1656	-0.8282	-3.000	5.8121	48.10	50.915	8.0	101.3	0.502	-31.1601	-1.9690
0.600	6.000	1.405	2.8910	11.2413	1.1564	-6.3977	-31.9883	-6.000	2.7015	9.72	6.419	8.0	101.3	0.063	-22.1933	-5.1702
0.720	0.000	1.468	-2.2792	11.7480												

t (sec)	0	0.12	0.24	0.36	0.48	0.60	0.72
v (in)	0	0.010	0.092	0.371	0.903	1.405	1.468

Problem 7-2

Following E 7-2: $\tilde{k}(t) = k(t) + \frac{6}{(\Delta t)^2} m + \frac{3}{\Delta t} c$

$$\tilde{k}(t) = k(t) + \frac{6}{(0.12)^2}(0.2) + \frac{3}{0.12}(0.4) = k(t) + 93.33$$

$$\left.\begin{aligned} \Delta\tilde{p}(t) &= \Delta p(t) + 11.2\,\dot{v}_0 + 0.624\,\ddot{v}_0 \\ \Delta\dot{v}(t) &= 25\,\Delta v - 3\dot{v}_0 - 0.06\ddot{v}_0 \end{aligned}\right\} \text{from P7-1}$$

COL. 1	COL. 2	COL. 3	COL. 4	COL. 5 (a)	COL. 6	COL. 7	COL. 8	COL. 9	COL.10	COL.11	COL.12	COL.13 (b)	COL.14	COL.15	COL.16	COL.17
0.000	0.000	0.000	0.0000	0.0000	0.0000	0.0000	0.0000	1.000	0.0000	0.00	1.000	8.0	101.3	0.010	4.1118	0.2467
0.120	1.000	0.010	0.2467	0.0789	0.0987	0.8224	4.1118	3.000	0.3947	4.93	8.329	8.0	101.3	0.082	9.5763	1.0680
0.240	4.000	0.092	1.3147	0.7365	0.5259	2.7376	13.6881	5.000	1.9062	21.36	28.266	8.0	101.3	0.279	9.4261	2.2081
0.360	9.000	0.371	3.5228	2.9680	1.4091	4.6228	23.1142	0.000	4.7822	49.10	53.879	8.0	101.3	0.532	-23.9424	1.3372
0.480	9.000	0.903	4.8600	7.2216	1.9440	-0.1656	-0.8282	-3.000	5.8121	48.10	50.915	8.0	101.3	0.502	-31.1601	-1.9690
0.600	6.000	1.405	2.8910	8.0000	1.1564	-3.1564	-15.7820	-6.000	3.0904	19.44	16.531	0.0	93.3	0.177	-23.4039	-3.2981
0.720	0.000	1.582	-0.4071	8.0000												

(a) $\bar{v} = v - v_i$, where v_i = inelastic displacement = $v_{max} - 1$ in.

(b) $k = 0$ while frame is yielding.

t (sec)	0.00	0.12	0.24	0.36	0.48	0.60	0.72
v (in)	0.000	0.010	0.092	0.371	0.903	1.405	1.582

Problem 7-3

$$\left(f_S\right)_0 = 8\left\{\left[1-\frac{1}{3}\left(\frac{2v}{3}\right)^2\right]v\right\}_0 \quad , \quad |v| < 1.5$$

eq. (7-22) : $k(t) = \left(\frac{df_S}{dv}\right)_0 = 8\left[1-\left(\frac{2v}{3}\right)^2\right]_0 \quad , \quad |v| < 1.5$

Following E7-2 :

$$\left.\begin{aligned} \tilde{k}(t) &= k(t) + 93.33 \\ \Delta\tilde{p}(t) &= \Delta p(t) + 11.2\,\dot{v}_0 + 0.624\,\ddot{v}_0 \\ \Delta\dot{v}(t) &= 25\,\Delta v - 3\dot{v}_0 - 0.06\,\ddot{v}_0 \end{aligned}\right\}\text{ from P7-2}$$

COL. 1	COL. 2	COL. 3	COL. 4	COL. 5	COL. 6	COL. 7	COL. 8	COL. 9	COL.10	COL.11	COL.12	COL.13	COL.14	COL.15	COL.16	COL.17
0.000	0.000	0.000	0.0000	0.0000	0.0000	0.0000	0.0000	1.000	0.0000	0.00	1.000	8.0	101.3	0.010	4.1118	0.2467
0.120	1.000	0.010	0.2467	0.0789	0.0987	0.8224	4.1118	3.000	0.3947	4.93	8.329	8.0	101.3	0.082	9.5764	1.0680
0.240	4.000	0.092	1.3147	0.7356	0.5259	2.7385	13.6927	5.000	1.9063	21.36	28.269	8.0	101.3	0.279	9.4586	2.2106
0.360	9.000	0.371	3.5254	2.9084	1.4101	4.6815	23.4075	0.000	4.7922	49.30	54.090	7.5	100.8	0.536	-22.9996	1.4289
0.480	9.000	0.907	4.9543	6.3742	1.9817	0.6441	3.2205	-3.000	6.0224	51.48	54.498	5.1	98.4	0.554	-26.6224	-1.2109
0.600	6.000	1.461	3.7434	7.9921	1.4974	-3.4894	-17.4472	-6.000	4.0733	26.97	25.039	0.4	.93.7	0.267	-23.5329	-3.5056
0.720	0.000	1.728	0.2378	8.0000												

t (sec)	0.00	0.12	0.24	0.36	0.48	0.60	0.72
v (in)	0.000	0.010	0.092	0.371	0.907	1.461	1.728

Problem 8-1

Since eq. of E 8-3c is equivalent to first eq. of sec. 8-4,

from $\omega^2 = k^*/m^*$ and $T = 2\pi/\omega \longrightarrow T = 2\pi\sqrt{m^*/k^*}$

$$T = 2\pi\sqrt{\frac{0.228\,\bar{m}L}{\dfrac{\pi^4 EI}{32L^3}}} = 1.720\,L^2\sqrt{\frac{\bar{m}}{EI}}$$

$$\therefore\ T = 1.720\,(200\,ft)^2\sqrt{\frac{110\ lb\cdot sec^2/ft^2}{165\times10^9\ lb\cdot ft^2}} = 1.776\ sec$$

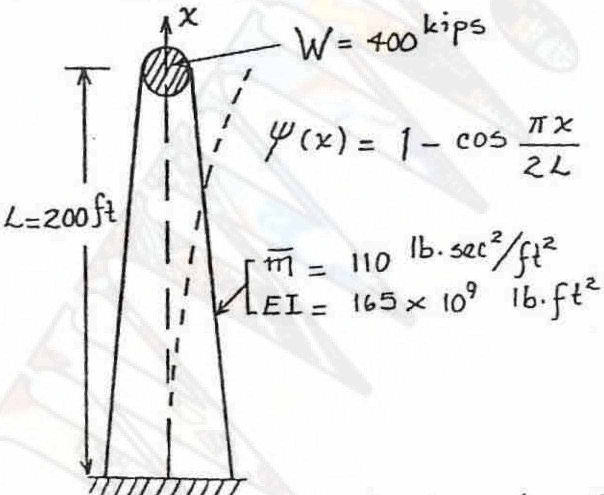
$$\blacktriangleright\ \underline{\underline{T = 1.78\ sec}}$$

More "accurate":

$$\left.\begin{aligned} m^* &= \left(\frac{3}{2}-\frac{4}{\pi}\right)\bar{m}L \\ k^* &= \frac{\pi^4}{32}\cdot\frac{EI}{L^3} \end{aligned}\right\} T = \frac{8L^2}{\pi}\sqrt{\left(3-\frac{8}{\pi}\right)\frac{\bar{m}}{EI}} = 1.771\ sec$$

$$\left.\begin{aligned} \bar{m} &= \frac{\gamma}{g}\,2\pi r t \\ I &= \pi t r^3 \end{aligned}\right\} T = \frac{16L}{\pi r}\sqrt{\left(\frac{3}{2}-\frac{4}{\pi}\right)\frac{L^2\gamma}{Eg}}$$

Problem 8-2



From eq. (8-18): $m^* = \int_0^L m(x)\left[\psi(x)\right]^2 dx + m\psi^2(L)$

From eq. of E8-3b: $m^* = \int_0^L m(x)\left[\psi(x)\right]^2 dx = 0.228\,\bar{m}L$

and $k^* = \int_0^L EI\left[\psi''(x)\right]^2 dx = \dfrac{\pi^4}{32}\dfrac{EI}{L^3}$

$$\therefore\ m^* = 0.228\,\bar{m}L + \frac{W}{g}(1)^2$$

(continued on following page)

Problem 8-2 (con'd)

$$m^* = 0.228\,(110\ ^{lb\cdot sec^2}/_{ft^2})(200ft) + \frac{400\times10^3\ lb}{32.22\ ft/sec^2} = 17430\ ^{lb\cdot sec^2}/_{ft}$$

$$k^* = \frac{\pi^4}{32}\,\frac{165\times10^9\ lb.ft^2}{(200\,ft)^3} = 62780\ ^{lb}/_{ft}$$

$$\therefore\ T = 2\pi\sqrt{\frac{m^*}{k^*}} = 2\pi\sqrt{\frac{17430}{62780}} = 3.311\ sec$$

$$\blacktriangleright\ \boxed{T = 3.31\ sec}$$

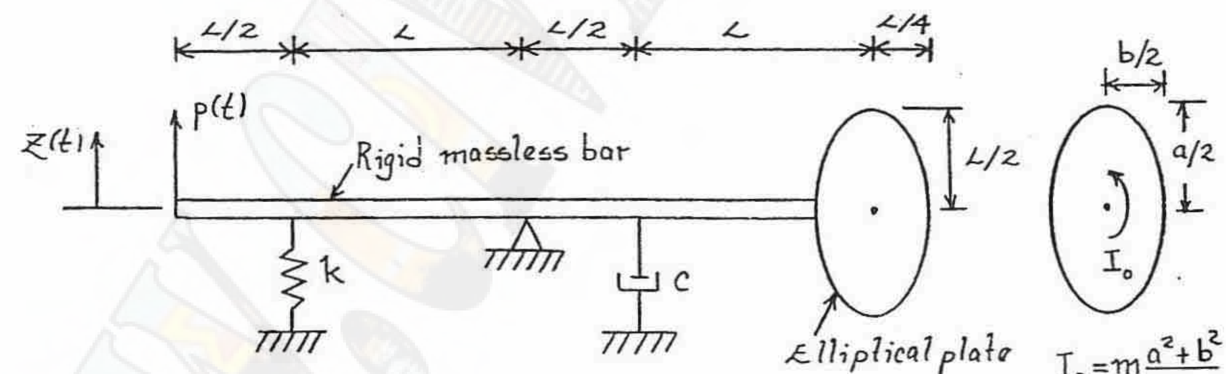
More "accurate":

$$m^* = \left(\frac{3}{2}-\frac{4}{\pi}\right)2\pi r t L\frac{\gamma}{g} + \frac{W}{g}$$

$$k^* = \frac{\pi^4}{32}\cdot\pi t r^3\cdot\frac{E}{L^3}$$

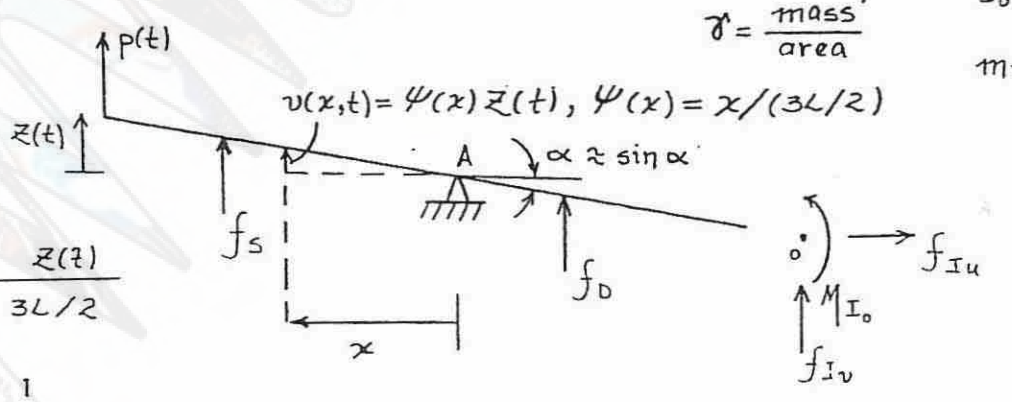
$$T = \frac{16L}{\pi r}\sqrt{\left[\left(\frac{3}{2}-\frac{4}{\pi}\right)L^2\gamma + \frac{WL}{2\pi r t}\right]\Big/(Eg)}$$

Problem 8-3



$$I_o = m\frac{a^2+b^2}{16}$$

$$m = \gamma\frac{\pi ab}{4}$$



$$\sin\alpha \approx \frac{Z(t)}{3L/2}$$

$$\cos\alpha \approx 1$$

(continued on following page)

Problem 8-3 (con'd)

$$m = \gamma \frac{\pi(L)(L/2)}{4} = \frac{\pi L^2}{8}\gamma$$

$$I_o = \frac{\pi L^2}{8}\gamma \, \frac{L^2 + (L/2)^2}{16} = \frac{5\pi L^4}{512}\gamma$$

Forces acting over the system:

$p(t)$

$$f_S = -k\, v(L,t) = -k \frac{L}{3L/2} z(t) = -\frac{2}{3} k\, z(t)$$

$$f_D = -c\, \dot{v}(-L/2,t) = -c \frac{-L/2}{3L/2}\dot{z}(t) = \frac{1}{3} c\, \dot{z}(t)$$

$$I_A \begin{cases} f_{Iv} = -m\, \ddot{v}(-3L/2, t) = -\gamma \dfrac{\pi(L)(L/2)}{4} \cdot \dfrac{-3L/2}{3L/2}\ddot{z}(t) = \gamma \dfrac{\pi L^2}{8}\ddot{z}(t) \\ f_{Iu} = m\, \ddot{u}(-3L/2, t) = m \dfrac{3L}{2}(1-\cos\alpha) \approx 0 \\ M_{I_o} = -I_o\, \ddot{\alpha} \end{cases}$$

$$I_A = I_o + m\left(\frac{3L}{2}\right)^2 = \gamma \frac{5\pi}{512} L^4 + \frac{\pi L^2}{8}\gamma \cdot \frac{9L^2}{4} = \gamma \frac{149}{512}\pi L^4$$

(i) Direct equilibrium:

$$\sum M_A = 0, \quad M_P = p(t)\left(\frac{3L}{2}\right)$$

$$M_{f_S} = f_S\,(L) = -\frac{2}{3} k\, z(t)\, L$$

$$M_{f_D} = f_D\,(-L/2) = -\frac{1}{3} c\, \dot{z}(t)\,(L/2)$$

$$M_{I_A} = -I_A\ddot{\alpha} = -\gamma \frac{149}{512}\pi L^4 \frac{\ddot{z}(t)}{3L/2} \begin{cases} M_{f_{Iv}} = f_{Iv}\,(-3L/2) = -\gamma \dfrac{\pi L^2}{8}\ddot{z}(t)\,(3L/2) \\ M_{I_o} = -I_o\,\ddot{\alpha} = -\gamma \dfrac{5\pi}{512} L^4 \dfrac{\ddot{z}(t)}{3L/2} \end{cases}$$

$$\frac{3L}{2} p(t) - \frac{2}{3} L k\, z(t) - \frac{1}{6} L c\, \dot{z}(t) - \gamma \frac{3\pi}{16}\ddot{z}(t) - \gamma \frac{5\pi L^3}{768}\ddot{z}(t) = 0$$

$$\frac{3L}{2} p(t) - \frac{2}{3} L k\, z(t) - \frac{1}{6} L c\, \dot{z}(t) - \gamma \frac{149}{768}\pi L^3 \ddot{z}(t) = 0$$

$$\frac{149\pi}{768}\gamma L^3 \ddot{z}(t) + \frac{L}{6} c\, \dot{z}(t) + \frac{2L}{3} k\, z(t) = \frac{3L}{2} p(t)$$

(continued on following page)

Problem 8-3 (con'd)

(iii) Hamilton's Principle :

The Hamilton's principle leads to

$$m^* \ddot{z}(t) + c^* \dot{z}(t) + \overline{k^*} z(t) = p^*(t) \quad \text{first eq. of sec. 8-4}$$

where :

eq. (8-18)

$$m^* = \int_0^L m(x) [\psi(x)]^2 dx + \sum_i m_i \psi_i^2 + \sum_i I_{o_i} (\psi_i')^2$$

$$m^* = \int_0^L m(x) [\psi(x)]^2 dx + \sum_i (I_{oi} + m_i r_i^2)(\psi_i')^2$$

$$c^* = \int_0^L c(x) [\psi(x)]^2 dx + a_1 \int_0^L EI(x) [\psi''(x)]^2 dx + \sum_i c_i \psi_i^2$$

$$\overline{k^*} = \int_0^L k(x) [\psi(x)]^2 dx + \int_0^L EI(x) [\psi''(x)]^2 dx + \sum_i k_i \psi_i^2 - \int_0^L N(x) [\psi'(x)]^2 dx$$

$$p^*(t) = \int_0^L p(x,t)\, \psi(x)\, dx + \sum_i p_i(t) \psi_i(x)$$

In this case :

$$m^* = m [\psi(-3L/2)]^2 + I_o [\psi'(-3L/2)]^2 = \frac{\pi L^2}{8} \gamma (1) + \frac{5\pi L^4}{512} \gamma \left(\frac{1}{3L/2}\right)^2$$

$$m^* = \frac{149\pi}{1152} L^2 \gamma$$

or

$$m^* = [I_o + m(-3L/2)^2][\psi'(3L/2)]^2 = \left(\frac{5\pi L^4}{512}\gamma + \frac{\pi L^2}{8}\gamma \cdot \frac{9L^2}{4}\right)\left(\frac{1}{3L/2}\right)^2$$

$$m^* = \frac{149\pi}{1152} L^2 \gamma$$

$$c^* = c [\psi(-L/2)^2]^2 = c(1/3)^2 = c/9$$

$$\overline{k^*} = k^* = k [\psi(L)]^2 = k(2/3)^2 = 4k/9$$

$$p^*(t) = p(t)\ \psi(3L/2) = p(t)(1)$$

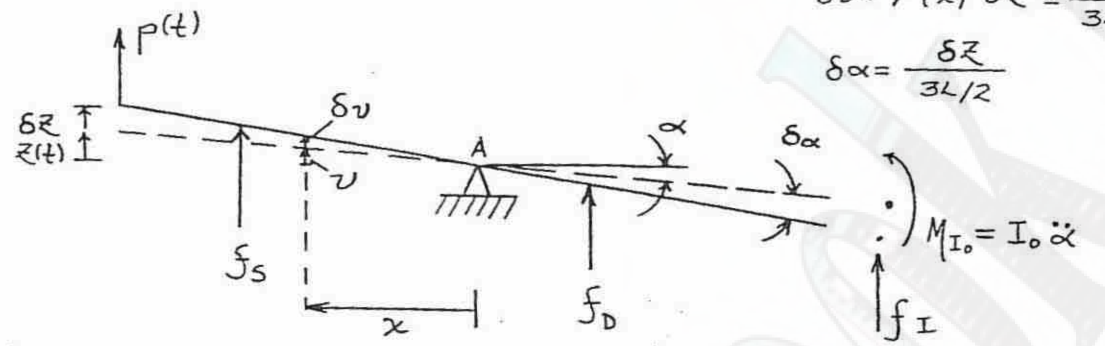
$$\blacktriangleright\ \boxed{m^* = \frac{149\pi}{1152} L^2 \gamma, \quad c^* = \frac{c}{9}, \quad k^* = \frac{4k}{9}, \quad p^*(t) = p(t)}$$

Problem 8-3 (con'd)

(ii) Virtual-Work analysis :

$$\delta v = \psi(x)\,\delta z = \frac{x}{3L/2}\,\delta z$$

$$\delta\alpha = \frac{\delta z}{3L/2}$$



$$\sum \delta W = 0 : \quad \delta W_p = p(t)\,\delta v(3L/2, t) = p(t)\,\frac{3L/2}{3L/2}\,\delta z$$

$$\delta W_{fs} = f_S\,\delta v(L, t) = -\frac{2}{3}k\,z(t)\,\frac{L}{3L/2}\,\delta z$$

$$\delta W_{fD} = f_D\,\delta v(-L/2, t) = \frac{c}{3}\ddot{z}(t)\,\frac{-L/2}{3L/2}\,\delta z \qquad \dot{z}_{(t)}$$

$$\delta W_{I_A} = -I_A\,\ddot{\alpha}\,\delta\alpha$$

$$\delta W_A = -\gamma\,\frac{149\pi L^5}{768}\ddot{z}(t)\frac{\delta z}{3L/2}$$

$$\delta W_{fI} = f_I\,\delta v(-3L/2, t) = \gamma\,\frac{\pi L^2}{8}\ddot{z}(t)\,\frac{-3L/2}{3L/2}\,\delta z$$

$$\delta W_{I_o} = -I_o\,\ddot{\alpha}\,\delta\alpha = -\gamma\,\frac{5\pi L^3}{768}\,\ddot{z}(t)\,\frac{\delta z}{3L/2}$$

$$0 = p(t)\delta z - \frac{4}{9}k\,z(t)\delta z - \frac{1}{9}c\,\dot{z}(t)\delta z - \gamma\frac{\pi L^2}{8}\ddot{z}(t)\delta z + \left(-\gamma\frac{5\pi L^2}{1152}\ddot{z}(t)\delta z\right)$$

$$0 = p(t)\,\delta z - \frac{4}{9}k\,z(t)\delta z - \frac{1}{9}c\,\dot{z}(t)\delta z - \gamma\,\frac{149\pi L^3}{1152}\ddot{z}(t)\delta z$$

Since $\delta z \neq 0$,

$$\frac{149\pi}{1152}L^3\gamma\,\ddot{z}(t) + \frac{c}{9}\dot{z}(t) + \frac{4}{9}k\,z(t) = p(t)$$

(continued on following page)

Problem 8-4 (con'd)

- Forces acting over the bar: $F = -\left[\frac{1}{2} m\, \ddot{z}(t) + k\, z(t)\right]$

$$f_D = -c\,\dot{v}(3L,t) = -c\,\dot{z}(t)$$

$$M_{I_{AB}} = -I_{AB}\,\ddot{\beta} = -\left[I_0 + m(3L/2)^2\right]\frac{\ddot{z}(t)}{3L}$$

$$M_{I_{AB}} = -\left(\frac{3mL^2}{4} + \frac{9mL^2}{4}\right)\frac{\ddot{z}(t)}{3L}$$

$$\bar{p}(t)$$

$\sum M_A = 0:$ $M_F = F(3L) = -\left[\frac{1}{2} m\, \ddot{z}(t) + k\, z(t)\right](3L)$

$$M_{f_D} = f_D\,(3L) = -c\,\dot{z}(t)\,(3L)$$

$$M_{I_{AB}} = -mL\,\ddot{z}(t)$$

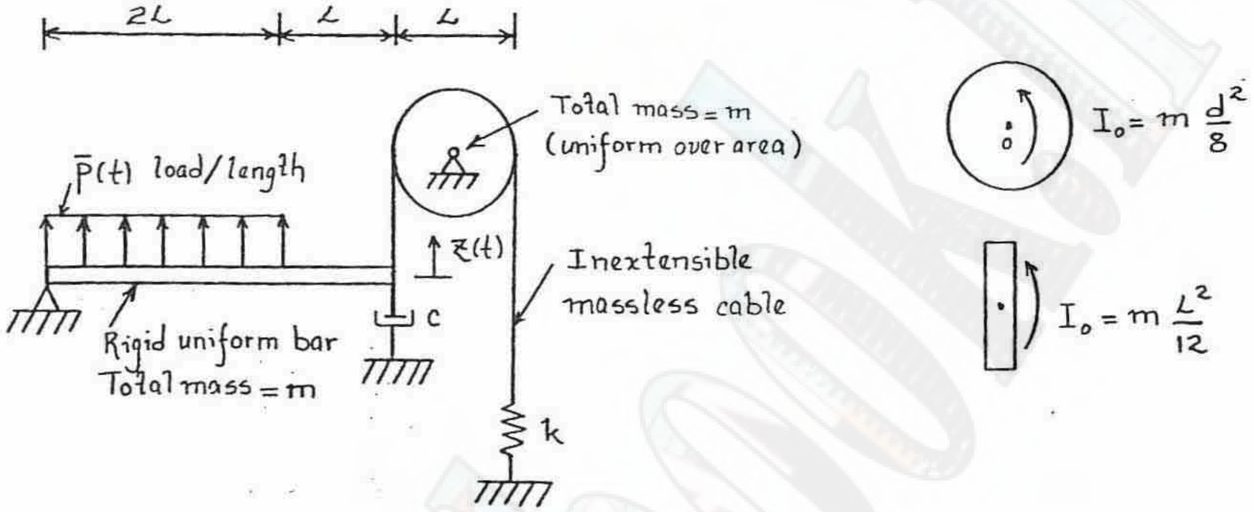
$$M_P = \int_0^{2L} \bar{p}(t)\, x\, dx = \bar{p}(t)\,\frac{1}{2}(4L^2)$$

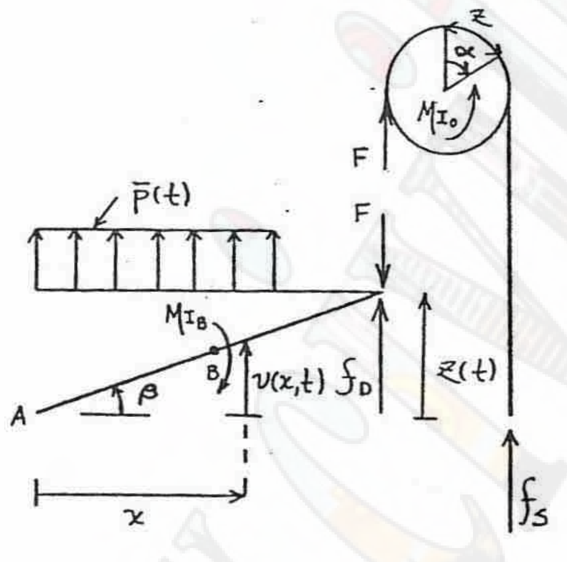
$$0 = -\frac{3}{2} mL\,\ddot{z}(t) - 3kL\,z(t) - 3cL\,\dot{z}(t) - mL\,\ddot{z}(t) + 2\,\bar{p}(t)\,L^2$$

$$\therefore\ \frac{5}{2} m\,\ddot{z}(t) + 3c\,\dot{z}(t) + 3k\,z(t) = 2\,\bar{p}(t)L$$

$$\blacktriangleright\ m^* = \frac{5}{4} m\,,\quad c^* = \frac{3}{2} c\,,\quad k^* = \frac{3}{2} k\,,\quad p^*(t) = \bar{p}(t)\,L$$

Problem 8-4





$$v(x,t) = \frac{x}{3L}\, z(t)$$

$$\alpha(t) = \frac{v(3L,t)}{L/2} = \frac{2}{L}\, z(t)$$

$$I_o = m\,\frac{L^2}{8} = \frac{mL^2}{8}$$

$$I_B = m\,\frac{(3L)^2}{12} = \frac{3mL^2}{4}$$

$$\beta = \frac{z(t)}{3L}$$

It is assumed the cable only transmits compression.

- Forces acting over the disk: $f_s = -k\, z(t)$

$$M_{I_o} = -I_o\,\ddot{\alpha} = -\frac{mL^2}{8}\,\frac{2}{L}\,\ddot{z}(t)$$

$$F = \; ?$$

$\sum M_o = 0:$

$$M_{f_s} = f_s\,(L/2) = -k\, z(t)\,(L/2)$$

$$M_{I_o} = -\frac{1}{4}\, m L\, \ddot{z}(t)$$

$$M_F = F\,(L/2)$$

$$0 = -k\, z(t)(L/2) - \frac{1}{4}\, mL\, \ddot{z}(t) + F(L/2)$$

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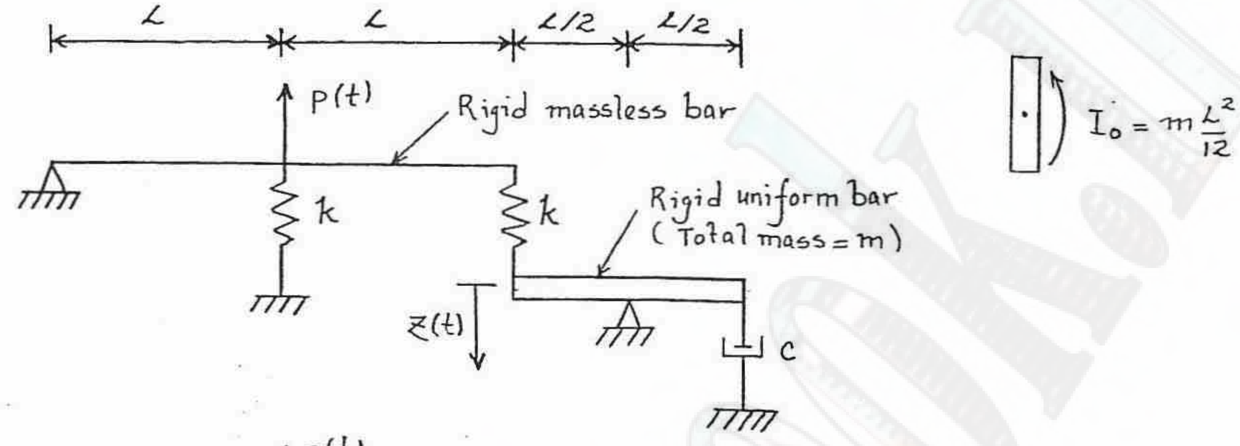
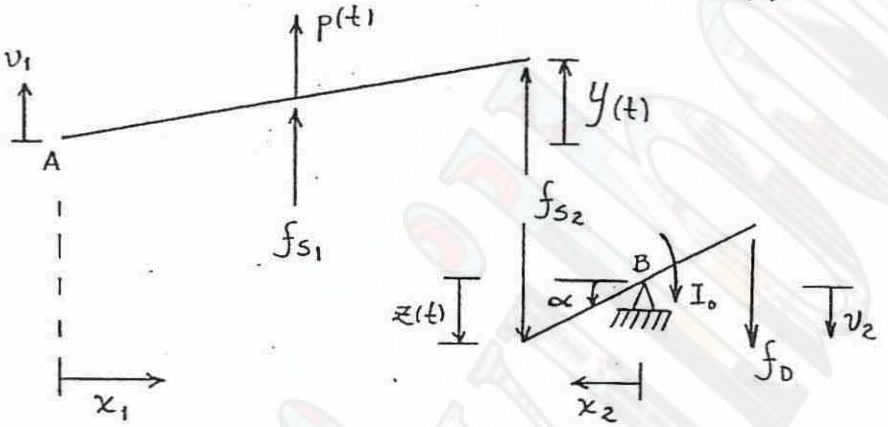
Problem 8-5





$$v_1 = \frac{x_1}{2L}\, y(t)$$

$$v_2 = \frac{x_2}{L/2}\, z(t)$$

$$\alpha = \frac{z(t)}{L/2}$$

. Forces acting over the system:

$$p(t)$$

$$f_{s_1} = -k\, v_1(x_1 = L,\, t) = -k\, \frac{1}{2}\, y(t)$$

$$f_{s_2} = -k\left[v_1(x_1 = 2L,\, t) + v_2(x_2 = L/2,\, t)\right] = -k\left[y(t) + z(t)\right]$$

$$f_D = -c\, \dot{v}_2(x_2 = -L/2,\, t) = -c\,(-1)\, \dot{z}(t)$$

$$I_o = m\, \frac{L^2}{12}$$

$$\sum M_A = 0: \quad M_p = p(t)\, L$$

$$M_{f_{s_1}} = f_{s_1}(L) = -k\, \frac{1}{2}\, y(t)\, L$$

$$M_{f_{s_2}} = f_{s_2}(2L) = -k\left[y(t) + z(t)\right](2L)$$

$$0 = p(t)L - \frac{1}{2}\, k\, y(t)L - 2k\left[y(t) + z(t)\right] L \qquad (1)$$

(continued on following page)

Problem 8-5 (con'd)

$$\sum M_B = 0: \quad M_{f_{S2}} = f_{S2}\,(L/2) = -k\,[y(t) + z(t)]\,(L/2)$$

$$M_{f_D} = f_D\,(-L/2) = c\,\dot{z}(t)\,(-L/2)$$

$$M_{I_0} = -I_0\,\ddot{\alpha} = -m\,\frac{L^2}{12}\cdot\frac{\ddot{z}(t)}{L/2}$$

$$0 = -\frac{1}{2}k\,[y(t) + z(t)]\,L - \frac{1}{2}c\,\dot{z}(t)\,L - \frac{1}{6}m\,\ddot{z}(t)\,L \qquad (2)$$

From (1): $y(t) = -\frac{4}{5}z(t) + \frac{2}{5}\frac{p(t)}{k}$

in (2): $0 = -\frac{2}{5}k\,z(t) + \frac{1}{5}p(t) + \frac{1}{2}k\,z(t) + \frac{5}{2}c\,\dot{z}(t) + \frac{5}{6}m\,\ddot{z}(t)$

$$\blacktriangleright\ m^* = \frac{5}{6}m, \quad c^* = \frac{5}{2}c, \quad k^* = \frac{1}{2}k, \quad p^*(t) = -p(t)$$

Problem 8-6

From eq. (8-14): $m^* = \int_0^L m(x)[\psi(x)]^2 dx = \int_0^L \bar{m}\left[\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)\right]^2 dx$

$$m^* = \bar{m}\int_0^L \left[\frac{9}{4}\left(\frac{x}{L}\right)^4 - \frac{3}{2}\left(\frac{x}{L}\right)^5 + \frac{1}{4}\left(\frac{x}{L}\right)^6\right] dx$$

$$m^* = \bar{m}\left[\frac{9}{4}\left(\frac{1}{5}L\right) - \frac{3}{2}\left(\frac{1}{6}L\right) + \frac{1}{4}\left(\frac{1}{7}L\right)\right] = \frac{33}{140}\,\bar{m}L$$

$$k^* = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx = \int_0^L EI\left[\frac{d}{dx}\left\{\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)\right\}\right]^2 dx = EI\int_0^L \left[\frac{3}{L^2} - 3\frac{x}{L^3}\right]^2 dx$$

$$k^* = EI\int_0^L \frac{9}{L^4}\left[1 - 2\frac{x}{L} + \left(\frac{x}{L}\right)^2\right] dx = \frac{9EI}{L^4}\left(L - L + \frac{1}{3}L\right) = \frac{3EI}{L^3}$$

From eq. (8-18): $p^*(t) = \int_0^L p(x,t)\,\psi(x)\,dx = \int_0^L \bar{p}(t)\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right) dx$

$$p^*(t) = \bar{p}(t)\int_0^L \frac{1}{2}\left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] dx = \frac{\bar{p}(t)}{2}\left(L - \frac{1}{4}L\right) = \frac{3}{8}\,\bar{p}(t)\,L$$

$$\blacktriangleright\ m^* = (33/140)\,\bar{m}L, \quad k^* = 3EI/L^3, \quad p^*(t) = (3/8)\,L\,\bar{p}(t)$$

Problem 8-8

(a) First eq. after eq. (8-20): $m^* = \int_0^a \int_0^b m(x,y)\left[\psi(x,y)\right]^2 dx\,dy$

$$m^* = \int_0^a \int_0^a \gamma\left(\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\right)^2 dx\,dy$$

$$m^* = \gamma\int_0^a \sin^2\frac{\pi x}{a}\,dx \int_0^a \sin^2\frac{\pi y}{a}\,dy = \gamma\left(\int_0^a \sin^2\frac{\pi x}{a}\,dx\right)^2$$

$$m^* = \gamma\left(\frac{a}{\pi}\int_0^a \sin^2\frac{\pi x}{a}\,d\left[\frac{\pi x}{a}\right]\right) = \gamma\left(\frac{a}{\pi}\right)^2\left[\left(\frac{\pi x}{2a} - \frac{\sin 2\frac{\pi x}{a}}{4}\right)\Bigg|_0^a\right]^2$$

$$m^* = \gamma\left(\frac{a}{\pi}\right)^2\left(\frac{\pi}{2}\right)^2 = \gamma\,\frac{a^2}{4}$$

Second eq. after eq. (8-20):

$$k^* = D\int_0^a\int_0^b\left\{\left[\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2}\right]^2 \right.$$
$$\left. -2(1-\nu)\left[\frac{\partial^2\psi(x,y)}{\partial x^2}\frac{\partial^2\psi(x,y)}{\partial y^2} - \left(\frac{\partial^2\psi(x,y)}{\partial x\partial y}\right)^2\right]\right\}dx\,dy$$

$$\psi(x,y) = \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$\therefore \frac{\partial\psi}{\partial x} = \frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} \longrightarrow \frac{\partial^2\psi}{\partial x^2} = -\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} = -\left(\frac{\pi}{a}\right)^2\psi(x,y)$$

$\dfrac{\partial^2\psi}{\partial x\partial y} = \left(\dfrac{\pi}{a}\right)^2\cos\dfrac{\pi x}{a}\cos\dfrac{\pi y}{a}$ in the same way, $\dfrac{\partial^2\psi}{\partial y^2} = -\left(\dfrac{\pi}{a}\right)^2\psi(x,y)$

$$\therefore \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2\left(\frac{\pi}{a}\right)^2\psi(x,y)$$

$$\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 = \left(\frac{\pi}{a}\right)^4\psi^2(x,y) - \left(\frac{\pi}{a}\right)^4\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}$$
$$= \left(\frac{\pi}{a}\right)^4\left[\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a} - \left(1-\sin^2\frac{\pi x}{a}\right)\left(1-\sin^2\frac{\pi y}{a}\right)\right]$$
$$= \left(\frac{\pi}{a}\right)^4\left(\sin^2\frac{\pi x}{a} + \sin^2\frac{\pi y}{a} - 1\right)$$

$$\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 = \left(\frac{\pi}{a}\right)^4\left(\sin^2\frac{\pi x}{a} - \cos^2\frac{\pi y}{a}\right)$$

(continued on following page)

Problem 8-7

$$\overline{k^*} = k^* - k_G^* \qquad \text{eq. (8-16)}$$

$$k^* = \frac{3EI}{L^3} \quad , \quad \text{from P8-6}$$

(a) From eq. (8-14): $k_G^* = N\int_0^L [\psi'(x)]^2 dx = N\int_0^L \left\{\frac{d}{dx}\left[\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)\right]\right\}^2 dx$

$$k_G^* = N\int_0^L \left(3\frac{x}{L^2} - \frac{3}{2}\frac{x^2}{L^3}\right)^2 dx$$

$$k_G^* = N\int_0^L \frac{9}{L^2}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 + \frac{1}{4}\left(\frac{x}{L}\right)^4\right] dx$$

$$k_G^* = \frac{9N}{L^2}\left(\frac{1}{3}L - \frac{1}{4}L + \frac{1}{20}L\right) = \frac{6}{5}\frac{N}{L}$$

$$\therefore \overline{k^*} = \frac{3EI}{L^3} - \frac{6}{5}\frac{N}{L} = \frac{3}{L}\left(\frac{EI}{L^2} - \frac{2}{5}N\right)$$

$$\blacktriangleright\ \overline{k^*} = \frac{3}{L}\left(\frac{EI}{L^2} - \frac{2}{5}N\right)$$

(b) From eq. (8-18): $k_G^* = \int_0^L N(x)[\psi'(x)]^2 dx$

$$k_G^* = \int_0^L N\left(1 - \frac{x}{L}\right)\frac{9}{L^2}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 + \frac{1}{4}\left(\frac{x}{L}\right)^4\right] dx$$

$$k_G^* = \frac{9N}{L^2}\int_0^L \left[\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 + \frac{5}{4}\left(\frac{x}{L}\right)^4 - \frac{1}{4}\left(\frac{x}{L}\right)^5\right] dx$$

$$k_G^* = \frac{9N}{L^2}\left(\frac{1}{3}L - \frac{1}{2}L + \frac{1}{4}L - \frac{1}{24}L\right) = \frac{3}{8}\frac{N}{L}$$

$$\therefore \overline{k^*} = \frac{3EI}{L^3} - \frac{3}{8}\frac{N}{L} = \frac{3}{L}\left(\frac{EI}{L^2} - \frac{1}{8}N\right)$$

$$\blacktriangleright\ \overline{k^*} = \frac{3}{L}\left(\frac{EI}{L^2} - \frac{N}{8}\right)$$

Problem 8-8 (con'd)

$$\therefore\ k^* = D\int_0^a\int_0^a \left\{\left[-2\left(\frac{\pi}{a}\right)^2 \psi(x,y)\right]^2 - 2(1-\nu)\left(\frac{\pi}{a}\right)^4\left(\sin^2\frac{\pi x}{a} - \cos^2\frac{\pi y}{a}\right)\right\} dx\,dy$$

$$k^* = D\left(\frac{\pi}{a}\right)^4 \left\{4\int_0^a\int_0^a \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\,dy\,dx - 2(1-\nu)\cdot\right.$$

$$\left.\left[\int_0^a\int_0^a \sin^2\frac{\pi x}{a}\,dy\,dx - \int_0^a\int_0^a \cos\frac{\pi y}{a}\,dx\,dy\right]\right\}$$

$$k^* = D\left(\frac{\pi}{a}\right)^4\left[4\cdot\frac{a^2}{4} - 2(1-\nu)\left(\int_0^a \sin^2\frac{\pi x}{a}\,dx\int_0^a dy - \int_0^a \cos^2\frac{\pi y}{a}\int_0^a dx\right)\right].$$

Since $\int_0^a \sin^2\frac{\pi x}{a}\,dx\int_0^a dy = \int_0^a \cos^2\frac{\pi y}{a}\int_0^a dx = \frac{\pi}{2}a$, then

$$k^* = \frac{\pi^4}{a^2}D$$

$$\blacktriangleright\ \boxed{m^* = \frac{1}{4}a^2\gamma\ ,\quad k^* = \pi^4\frac{D}{a^2}}$$

(b) Third eq. after eq. (8-20): $p^*(t) = \int_0^a\int_0^b p(x,y)\,\psi(x,y)\,dx\,dy$

$$p^*(t) = \int_0^a\int_0^a \bar{p}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\,dx\,dy$$

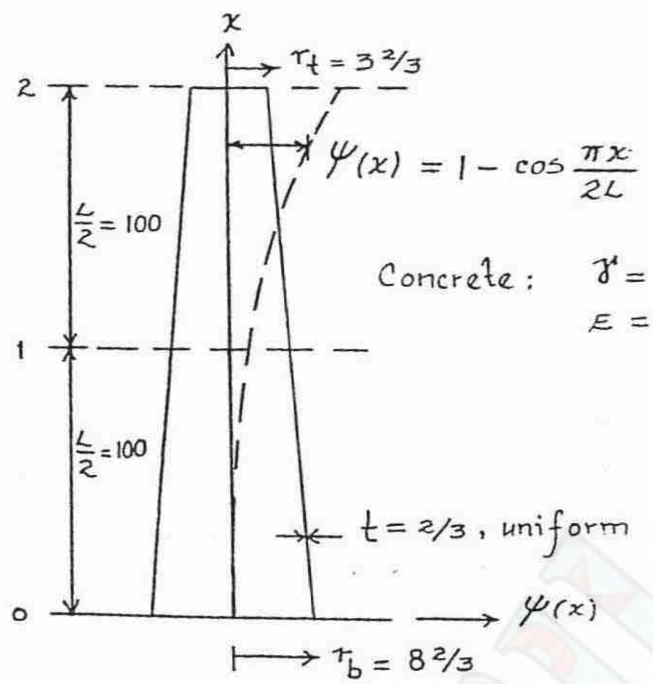
$$p^*(t) = \bar{p}(t)\int_0^a \sin\frac{\pi x}{a}dx\int_0^a \sin\frac{\pi y}{a}\,dy = \bar{p}(t)\left[\int_0^a \sin\frac{\pi x}{a}\,dx\right]_0^2$$

$$p^*(t) = \bar{p}(t)\left[\frac{a}{\pi}\int_0^a \sin\frac{\pi x}{a}\,d\left(\frac{\pi x}{a}\right)\right]^2 = \bar{p}(t)\left(\frac{a}{\pi}\right)^2\left[\left(-\cos\frac{\pi x}{a}\right)\Big|_0^a\right]^2$$

$$p^*(t) = \bar{p}(t)\left(\frac{a}{\pi}\right)^2(-1-1)^2 = \frac{4a^2}{\pi^2}\bar{p}(t)$$

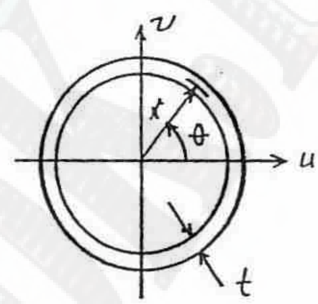
$$\blacktriangleright\ \boxed{p^*(t) = \frac{4}{\pi^2}a^2\bar{p}(t)}$$

Problem 8-9



Concrete: $\gamma = 0.150$
$E = 0.436 \times 10^6$

- Units : kips-ft
- $g = 32.2$ ft/sec



$$\text{at } x:\ r(x) = (r_t - r_b)\frac{x}{L} + r_b$$

$$m(x) = \rho A(x) = \frac{\gamma}{g}\left[2\pi t\, r(x)\right]$$

$$I_u(x) = I_v(x) = \pi t r^3(x)\left\{1 + \left[\frac{t}{2r(x)}\right]^2\right\}$$

$$I_{approx}(x) \approx \pi t r^3(x)$$

- Simpson's Rule:

i	x_i (ft)	$r_i = (r_t - r_b)\frac{x_i}{L} + r_b$ (ft)	$\overline{m}_i = 2\pi t r_i \frac{\gamma}{g}$ (kips·sec²/ft²)	$\psi_i = 1 - \cos\frac{\pi x_i}{2L}$	$\tilde{m}_i = \overline{m}_i \psi_i^2$ (kips·sec²/ft²)	$EI_i = \pi t r_i^3 E$ (kips-ft²)	$\psi_i'' = \left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x_i}{2L}$ (1/ft²)	$\tilde{k}_i = EI_i(\psi_i'')^2$ (kips/ft²)
0	0	8.6667	—	0	0	0.59444×10^9	6.1685×10^{-5}	2.26190
1	100	6.1667	0.120331	0.29289	0.103225	0.21414×10^9	4.3612×10^{-5}	0.40730
2	200	3.6667	0.071548	1	0.071548	—	0	0

$$m^* = \frac{L}{6}(\tilde{m}_0 + 4\tilde{m}_1 + \tilde{m}_2) = 3.76 \text{ kips·sec}^2/\text{ft} \qquad k^* = \frac{L}{6}(\tilde{k}_0 + 4\tilde{k}_1 + \tilde{k}_2) = 129.7 \text{ kips/ft}$$

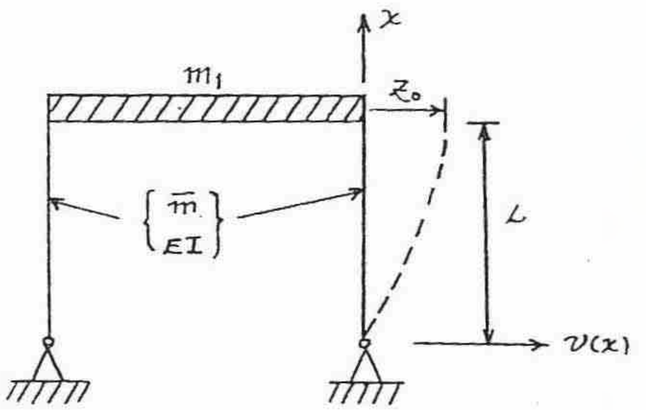
- "Exact":

$$m^* = \int_0^L m(x)\psi^2(x)\,dx = \int_0^L \left[2\pi t\frac{\gamma}{g}r(x)\right]\left[1 - \cos\frac{\pi x}{2L}\right]^2 dx = 2\pi t r_b \frac{\gamma}{g}\left(0.040752 + 0.186008\frac{r_t}{r_b}\right) = 4.04 \frac{\text{kips·sec}^2}{\text{ft}}$$

$$k^* = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx = \int_0^L \left\{\pi t E\left[r^3(x) + \frac{t^2}{4}r(x)\right]\right\}\left\{\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L}\right\}^2 dx$$

$$k^* = \left(\frac{\pi}{2}\right)^5\left(\frac{r_b}{L}\right)^3 t E\left[0.43077 + 0.31561\frac{r_t}{r_b} + 0.184388\left(\frac{r_t}{r_b}\right)^2 + 0.069228\left(\frac{r_t}{r_b}\right)^3 + \left(\frac{t}{2r_b}\right)^2\left(0.70264 + 0.29736\frac{r_t}{r_b}\right)\right] = 136.7 \text{ kips/ft} \quad (= 136.3, \text{ using } I_{approx})$$

Problem 8-11



$$v(x) = \frac{PL^3}{12EI}\left[3\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

eq. (8-25): $v(x,t) = \psi(x)\, Z_0 \sin \omega t$

$$\therefore\ \psi(x) = 3\frac{x}{L} - \left(\frac{x}{L}\right)^3$$

eq. (8-27): $$V_{max} = \frac{1}{2} Z_0^2 \int_0^L EI(x)\left[\psi''(x)\right]^2 dx$$

$$V_{max} = \frac{1}{2} Z_0^2\, 2\int_0^L EI\left\{\frac{d^2}{dx^2}\left[3\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]\right\}^2 dx$$

$$V_{max} = Z_0^2\, EI\int_0^L 36\,\frac{x^2}{L^6}\,dx = Z_0^2\, EI\,\frac{36}{L^6}\cdot\frac{1}{3}L^3 = Z_0^2\,\frac{12EI}{L^3} \quad (*)$$

From eq. (8-29): $$T_{max} = \frac{1}{2} Z_0^2 \omega^2 \int_0^L m(x)\left[\psi(x)\right]^2 dx$$

and eq. (8-18): $$m^* = \int_0^L m(x)\left[\psi(x)\right]^2 dx + \sum m_i \psi_i^2 + \sum I_{oi}(\psi_i')^2$$

$$T_{max} = \frac{1}{2} Z_0^2 \omega^2\left\{2\int_0^L \bar{m}\left[3\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]^2 dx + m_1\left[3(1) - 1^3\right]^2\right\}$$

$$T_{max} = \frac{1}{2} Z_0^2 \omega^2\left\{2\bar{m}\int_0^L\left[9\left(\frac{x}{L}\right)^2 - 6\left(\frac{x}{L}\right)^4 + \left(\frac{x}{L}\right)^6\right]dx + 4m_1\right\}$$

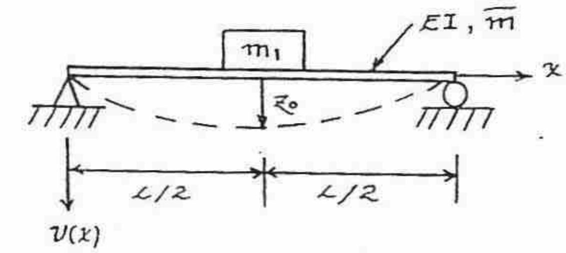
$$T_{max} = \frac{1}{2} Z_0^2 \omega^2\left[2\bar{m}\left(3L - \frac{6}{5}L + \frac{1}{7}L\right) + 4m_1\right]$$

$$T_{max} = \frac{1}{2} Z_0^2 \omega^2\left[2\left(\frac{68}{35}\right)\bar{m}L + 4m_1\right] \quad \binom{*}{*}$$

Equating $(*)$ and $\binom{*}{*}$: $$\omega^2 = \frac{24EI}{L^3\left[2\left(\frac{68}{35}\right)\bar{m}L + 4m_1\right]} = \frac{6EI}{L^3\left(\frac{34}{35}\bar{m}L + m_1\right)}$$

(continued on following page)

Problem 8-10



$$v(x)=\begin{cases} Px(3L^2-4x^2)/(48EI), & 0\le x\le \frac{L}{2} \\ \text{symmetric respect to } x=L/2 \end{cases}$$

Let $\quad v(x,t)=\psi(x)\,Z_0\sin\omega t \qquad$ eq. (8-25)

$$\therefore \psi(x)=\frac{x}{L}\left[3-4\left(\frac{x}{L}\right)^2\right]$$

From eq. (8-27): $\quad V_{max}=\frac{1}{2}Z_0^2\int_0^L EI(x)\left[\psi''(x)\right]^2dx$

$$V_{max}=\frac{1}{2}Z_0^2\cdot 2\int_0^{L/2}EI\left(\frac{d^2}{dx^2}\left\{\frac{x}{L}\left[3-4\left(\frac{x}{L}\right)^2\right]\right\}\right)^2dx$$

$$V_{max}=Z_0^2\,EI\int_0^{L/2}576\frac{x^2}{L^6}dx=Z_0^2\,EI\,\frac{576}{L^6}\,\frac{1}{3}\left(\frac{L}{2}\right)^3=Z_0^2\,\frac{24EI}{L^3} \qquad (*)$$

From eqs. (8-29) and (8-18):

$$T_{max}=\frac{1}{2}Z_0^2\omega^2\left(2\int_0^{L/2}\bar{m}\left\{\frac{x}{L}\left[3-4\left(\frac{x}{L}\right)^2\right]\right\}^2dx+m_1\left\{\frac{L/2}{L}\left[3-4\left(\frac{L/2}{L}\right)^2\right]\right\}^2\right)$$

$$T_{max}=\frac{1}{2}Z_0^2\omega^2\left\{2\bar{m}\int_0^{L/2}\left[9\left(\frac{x}{L}\right)^2-24\left(\frac{x}{L}\right)^4+16\left(\frac{x}{L}\right)^6\right]dx+m_1(1)^2\right\}$$

$$T_{max}=\frac{1}{2}Z_0^2\omega^2\left[2\bar{m}\left(3\cdot\frac{L}{8}-\frac{24}{5}\cdot\frac{L}{32}+\frac{16}{7}\cdot\frac{L}{128}\right)+m_1\right]=\frac{1}{2}Z_0^2\omega^2\left(\frac{17}{35}\bar{m}L+m_1\right) \qquad (**)$$

Equating $(*)$ and $(**)$: $\quad \omega^2=\dfrac{48EI}{\left(\frac{17}{35}\bar{m}L+m_1\right)L^3}$

$$\therefore\ T=\frac{\pi}{2}\sqrt{\frac{\left(\frac{17}{35}\bar{m}L+m_1\right)L^3}{3EI}}$$

(a)

$$\blacktriangleright\ T=\frac{\pi}{2}\sqrt{\frac{17\bar{m}L^4}{105EI}}$$

(b)

$$\blacktriangleright\ T=\frac{\pi}{2}\sqrt{\frac{\left(\frac{17}{35}\bar{m}L+3\bar{m}L\right)L^3}{3EI}}=\frac{\pi}{2}\sqrt{\frac{122\bar{m}L^4}{105EI}}$$

Problem 8-11 (con'd)

$$\therefore\ T = 2\pi\sqrt{\frac{L^3\left(\frac{34}{35}\bar{m}L + m_1\right)}{6EI}}$$

(a) $$T = 2\pi\sqrt{\frac{L^3\left(\frac{34}{35}\bar{m}L + 4\bar{m}L\right)}{6EI}} = 2\pi\sqrt{\frac{29}{35}\cdot\frac{\bar{m}L^4}{EI}}$$

$$\blacktriangleright\ \boxed{T = 2\pi\sqrt{\frac{29}{35}\cdot\frac{\bar{m}L^4}{EI}}}$$

(b) if $\frac{19}{3}\bar{m}L \equiv 0$ in $T = 2\pi\sqrt{\frac{L^3\left(\frac{34}{35}\bar{m}L + m_1\right)}{6EI}}$, *i.e.,* no distributed mass in columns and $m_1 = 4\bar{m}L + \alpha m_T$ = girder mass + $\alpha\times$(total column mass), *i.e.,* column weight lumped with girder, where $m_T = 2\bar{m}L$

$$\therefore\ T = 2\pi\sqrt{\frac{L^3\left[0 + (4\bar{m}L + \alpha\, 2\bar{m}L)\right]}{6EI}} \equiv 2\pi\sqrt{\frac{29}{35}\cdot\frac{\bar{m}L^4}{EI}}$$

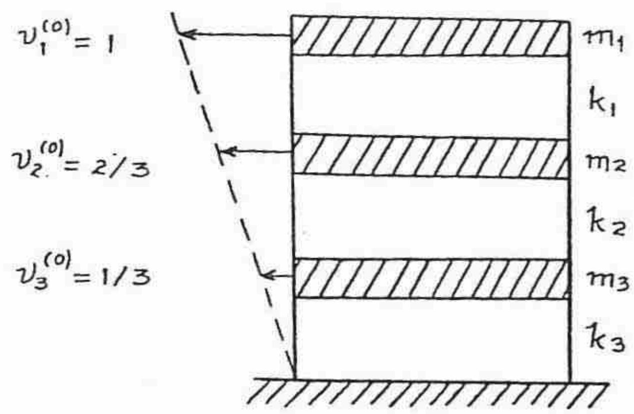
$$2\pi\sqrt{\frac{2+\alpha}{3}\cdot\frac{\bar{m}L^4}{EI}} \equiv 2\pi\sqrt{\frac{29}{35}\cdot\frac{\bar{m}L^4}{EI}}$$

$$\therefore\quad \frac{2+\alpha}{3} = \frac{29}{35}$$

$$\alpha = \frac{17}{35}$$

$$\blacktriangleright\ \boxed{\alpha = \frac{17}{35} = 48.6\%}$$

Problem 8-12



$m_1 = m_2 = m_3 = m = 2 \ \text{kips}\cdot\text{sec}^2/\text{in}$

$k_1 = \frac{1}{2} k_2 = \frac{1}{3} k_3 = k = 400 \ \text{kips/in}$

Let $v_i^{(0)}(t) = \psi_i^{(0)} Z_0^{(0)} \sin \omega t$, eq. (8-33)

then $\psi_1^{(0)} = 1$, $\psi_2^{(0)} = 2/3$, $\psi_3^{(0)} = 1/3$, $Z_0^{(0)} = 1$

Following E8-6:

(a) $T_{max}^{(0)} = \frac{1}{2} \omega^2 \left[Z_0^{(0)} \right]^2 \sum_{i=1}^{3} m_i \left[\psi_i^{(0)} \right]^2 = \frac{1}{2} \omega^2 \left[Z_0^{(0)} \right]^2 m \left(1 + \frac{4}{9} + \frac{1}{9} \right)$

$T_{max}^{(0)} = \frac{1}{2} \left[Z_0^{(0)} \right]^2 \omega^2 \ \frac{14 m}{9}$ $\qquad (*)$

$V_{max}^{(0)} = \frac{1}{2} \left[Z_0^{(0)} \right]^2 \sum_{i=1}^{3} k_i \left[\Delta \psi_i^{(0)} \right]^2 = \frac{1}{2} \left[Z_0^{(0)} \right]^2 \left(k \frac{1}{9} + 2k \frac{1}{9} + 3k \frac{1}{9} \right)$

$V_{max}^{(0)} = \frac{1}{2} \left[Z_0^{(0)} \right]^2 \frac{6k}{9}$ $\qquad \left({*\atop *} \right)$

Equating $(*)$ and $\left({*\atop *} \right)$: $\qquad \omega_{R00}^2 = \frac{3k}{m}$

$$T_{R00} = 2\pi \sqrt{\frac{7m}{3k}} = 0.679 \text{ sec}$$

▶ $\boxed{T_{R00} = 0.679 \text{ sec}}$

Loading the structure with the inertial forces given by eq. (8-37):

$p_i^{(0)} = \omega^2 m_i v_i^{(0)} = Z_0^{(0)} \omega^2 m_i \psi_i^{(0)}$

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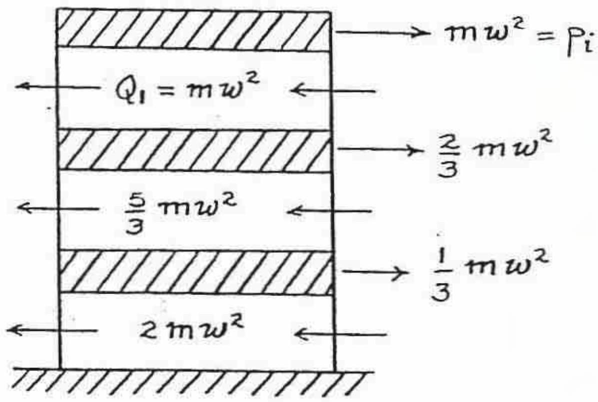
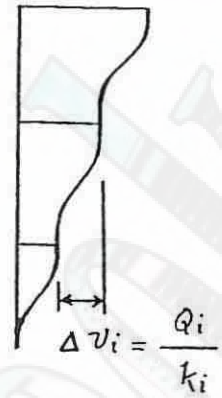
Problem 8-12 (con'd)





$$\Delta v_1 = \frac{m w^2}{k}$$

$$\Delta v_2 = \frac{\frac{5}{3} m w^2}{2k}$$

$$\Delta v_3 = \frac{2 m w^2}{3k}$$

From eq. (8-38): $v_i^{(1)} = w^2 \psi_i^{(1)} \bar{Z}_0^{(1)}$ and knowing that $v_i = \sum_{j=3}^{i} \Delta v_j$

$$\psi_1^{(1)} = 1, \quad \psi_2^{(1)} = \frac{3}{5}, \quad \psi_3^{(1)} = \frac{4}{15}, \quad \bar{Z}_0^{(1)} = \frac{5}{2}\frac{m}{k}$$

(b) $$V_{max} = \frac{1}{2} w^4 Z_0^{(0)} \bar{Z}_0^{(1)} \sum_{i=1}^{3} m_i \psi_i^{(0)} \psi_i^{(1)} = \frac{1}{2} w^4 Z_0^{(0)} \bar{Z}_0^{(1)} m \left(1 + \frac{2}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot\frac{4}{15}\right)$$

$$V_{max} = \frac{1}{2} Z_0^{(0)} \bar{Z}_0^{(1)} w^4 \frac{67}{45} m \qquad \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

Equating $(*)$ and $\begin{pmatrix} * \\ * \\ * \end{pmatrix}$: $$w_{R01}^2 = \frac{Z_0^{(0)}}{\bar{Z}_0^{(1)}} \frac{70}{67} = \frac{28}{67}\frac{k}{m}$$

then $$T_{R01} = 2\pi \sqrt{\frac{67}{28}\frac{m}{k}} = 0.687 \text{ sec}$$

▶ $\underline{\underline{T_{R01} = 0.687 \text{ sec}}}$

(c) $$T_{max}^{(1)} = \frac{1}{2} w^6 \left[\bar{Z}_0^{(1)}\right]^2 \sum_{i=1}^{3} m_i \left[\psi_i^{(1)}\right]^2 = \frac{1}{2} w^6 \left[Z_0^{(1)}\right]^2 m \left(1 + \frac{9}{25} + \frac{16}{225}\right)$$

$$T_{max}^{(1)} = \frac{1}{2} w^6 \left[\bar{Z}_0^{(1)}\right]^2 \frac{322}{225} m \qquad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

Equating $\begin{pmatrix} * \\ * \\ * \end{pmatrix}$ and $\begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$: $$w_{R11}^2 = \frac{Z_0^{(0)}}{\bar{Z}_0^{(1)}} \cdot \frac{335}{322} = \frac{67}{161}\frac{k}{m}$$

$$\therefore T_{R11} = 2\pi \sqrt{\frac{161}{67}\frac{m}{k}} = 0.689 \text{ sec}$$

▶ $\underline{\underline{T_{R11} = 0.689 \text{ sec}}}$

Problem 8-13

Following P8-12, but for this case, $m_1 = \frac{1}{2} m_2 = \frac{1}{3} m_3 = 1 \ \mathrm{kips \cdot sec^2/in} = m$

$k_1 = k_2 = k_3 = 800 \ \mathrm{kips/in} = k$

(a) $$T_{max}^{(0)} = \frac{1}{2} w^2 \left[Z_0^{(0)}\right]^2 \left[m(1) + 2m\left(\frac{4}{9}\right) + 3m\left(\frac{1}{9}\right)\right]$$

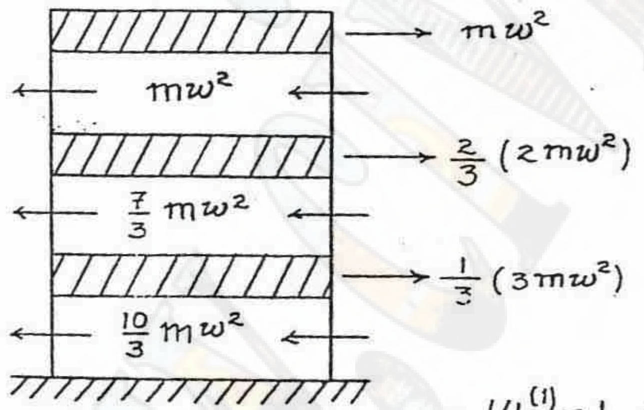
$$T_{max}^{(0)} = \frac{1}{2} w^2 \left[Z_0^{(0)}\right]^2 \frac{20}{9} m \qquad (*)$$

$$V_{max}^{(0)} = \frac{1}{2} \left[Z_0^{(0)}\right]^2 k\left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{1}{2}\left[Z_0^{(0)}\right]^2 \frac{1}{3} k \qquad \binom{*}{*}$$

Equating $(*)$ and $\binom{*}{*}$: $\quad w_{R00}^2 = \frac{3}{20} \frac{k}{m}$

$$\therefore T_{R00} = 4\pi \sqrt{\frac{5m}{3k}} = 0.574 \text{ sec}$$

► $T_{R00} = 0.574$ sec



$\Delta v_1 = \frac{mw^2}{k} \qquad v_1 = w^2 \frac{20}{3}\frac{m}{k} = w^2 \bar{Z}_0^{(1)} \psi_1^{(1)}$

$\Delta v_2 = \frac{\frac{7}{3} mw^2}{k} \qquad v_2 = w^2 \frac{17}{3} \frac{m}{k}$

$\Delta v_3 = \frac{\frac{10}{3} mw^2}{k} \rightarrow v_3 = w^2 \frac{10}{3}\frac{m}{k}$

$$\therefore \psi_1^{(1)} = 1, \ \psi_2^{(1)} = \frac{17}{20}, \ \psi_3^{(1)} = \frac{1}{2}, \ \bar{Z}_0^{(1)} = \frac{20m}{3k}$$

(continued on following page)

Problem 8-13 (con'd)

(b) $$V_{max}^{(1)} = \frac{1}{2}\omega^4 Z_0^{(0)} \bar{Z}_0^{(1)}\left[m(1) + 2m\left(\frac{2}{3}\right)\left(\frac{17}{20}\right) + 3m\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\right]$$

$$V_{max}^{(1)} = \frac{1}{2}\omega^4 Z_0^{(0)} \bar{Z}_0^{(1)} \frac{79}{30} m \qquad \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

Equating $(*)$ and $\begin{pmatrix} * \\ * \\ * \end{pmatrix}$: $$\omega_{R01}^2 = \frac{Z_0^{(0)}}{\bar{Z}_0^{(1)}}\frac{200}{237} = \frac{1}{\frac{20m}{3k}}\cdot\frac{200}{237} = \frac{10}{79}\frac{k}{m}$$

$$\therefore \; T_{R01} = 2\pi\sqrt{7.9\frac{m}{k}} = 0.624 \text{ sec}$$

► $\boxed{T_{R01} = 0.624 \text{ sec}}$

(c) $$T_{max}^{(1)} = \frac{1}{2}\omega^6\left[\bar{Z}_0^{(1)}\right]^2\left(m + 2m\frac{289}{400} + 3m\frac{1}{4}\right)$$

$$T_{max}^{(1)} = \frac{1}{2}\omega^6\left[\bar{Z}_0^{(1)}\right]^2\frac{639}{200} m \qquad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

Equating $\begin{pmatrix} * \\ * \\ * \end{pmatrix}$ and $\begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$: $$\omega_{R11}^2 = \frac{Z_0^{(0)}}{\bar{Z}_0^{(1)}}\frac{79(20)}{3(639)} = \frac{1}{\frac{20m}{3k}}\cdot\frac{79(20)}{3(639)}$$

$$\omega_{R11}^2 = \frac{79}{639}\frac{k}{m}$$

$$\therefore \; T_{R11} = 2\pi\sqrt{\frac{639}{79}\frac{m}{k}} = 0.632 \text{ sec}$$

► $\boxed{T_{R11} = 0.632 \text{ sec}}$

Problem 10-1

Eq. (10-21): $k_{ij} = \int_0^L EI(x)\,\psi_i''(x)\,\psi_j''(x)\,dx$

$$\therefore\ k_{23} = \int_0^L EI(x)\,\psi_2''(x)\,\psi_3''(x)\,dx$$

Eq. (10-16c): $\psi_2(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \longrightarrow \psi_2''(x) = 6\frac{1}{L^2} - 12\frac{x}{L^3}$

Eq. (10-16b): $\psi_3(x) = x\left(1-\frac{x}{L}\right)^2 \longrightarrow \psi_3''(x) = -4\frac{1}{L} + 6\frac{x}{L^2}$

$$\therefore k_{23} = \int_0^L EI_0\left(1+\frac{x}{L}\right)\frac{6}{L^2}\left(1-2\frac{x}{L}\right)\frac{2}{L}\left(3\frac{x}{L}-2\right)dx$$

$$k_{23} = \frac{12EI_0}{L^3}\int_0^L\left[-2+5\frac{x}{L}+\left(\frac{x}{L}\right)^2-6\left(\frac{x}{L}\right)^3\right]dx$$

$$k_{23} = \frac{12EI_0}{L^3}\left(-2+\frac{5}{2}+\frac{1}{3}-\frac{6}{4}\right)L = -\frac{8EI_0}{L^2}$$

$$\blacktriangleright\ k_{23} = -\frac{8EI_0}{L^2}$$

Problem 10-2

Eq. (10-28): $m_{ij} = \int_0^L m(x)\,\psi_i(x)\,\psi_j(x)\,dx$

$$\therefore\ m_{23} = \int_0^L m(x)\,\psi_2(x)\,\psi_3(x)\,dx$$

Eq. (10-16c): $\psi_2(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$, and eq. (10-16b): $\psi_3(x) = x\left(1-\frac{x}{L}\right)^2$

$$\therefore m_{23} = \int_0^L \bar{m}\left(1+\frac{x}{L}\right)\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right]x\left(1-\frac{x}{L}\right)^2dx$$

$$m_{23} = \bar{m}\int_0^L\left[3\left(\frac{x}{L}\right)^2-5\left(\frac{x}{L}\right)^3-\left(\frac{x}{L}\right)^4+5\left(\frac{x}{L}\right)^5-2\left(\frac{x}{L}\right)^6\right]x\,dx = \bar{m}\left(\frac{3}{4}-1-\frac{1}{6}+\frac{5}{7}-\frac{1}{4}\right)L^2$$

$$\blacktriangleright\ m_{23} = \frac{\bar{m}L^2}{84}(4) = \frac{1}{21}\bar{m}L^2$$

Problem 10-3

Eq. (10-34a): $p_i(t) = f(t)\int_0^L \chi(x)\,\psi_i(x)\,dx$

$\therefore \quad p_2(t) = f(t)\int_0^L \chi(x)\,\psi_2(x)\,dx$

Eq. (10-34): $p(x,t) = \chi(x)\,f(t) = \bar{p}\left(2+\frac{x}{L}\right)\sin\bar{\omega}t \longrightarrow \begin{cases}\chi(x) = 2+\frac{x}{L}\\ f(t) = \bar{p}\sin\bar{\omega}t\end{cases}$

Eq. (10-16c): $\psi_2(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$

$$\therefore p_2(t) = \bar{p}\sin\bar{\omega}t\int_0^L\left(2+\frac{x}{L}\right)\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]dx$$

$$p_2(t) = \bar{p}\sin\bar{\omega}t\int_0^L\left[6\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 - 2\left(\frac{x}{L}\right)^4\right]dx$$

$$p_2(t) = \bar{p}\sin\bar{\omega}t\left(2-\frac{1}{4}-\frac{2}{5}\right)L = \frac{27}{20}\bar{p}L\sin\bar{\omega}t$$

► $\boxed{p_2(t) = \frac{27}{20}\bar{p}L\sin\bar{\omega}t}$

Problem 10-4

Eq. (10-42): $k_{Gij} = \int_0^L N(x)\,\psi_i'(x)\,\psi_j'(x)\,dx$

$\therefore\ k_{G24} = \int_0^L N(x)\,\psi_2'(x)\,\psi_4'(x)\,dx$

Eq. (10-16c): $\psi_2(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \longrightarrow \psi_2'(x) = 6\frac{x}{L^2} - 6\frac{x^2}{L^3}$

Eq. (10-16d): $\psi_4(x) = \frac{x^2}{L}\left(\frac{x}{L}-1\right) \longrightarrow \psi_4'(x) = 3\left(\frac{x}{L}\right)^2 - 2\frac{x}{L}$

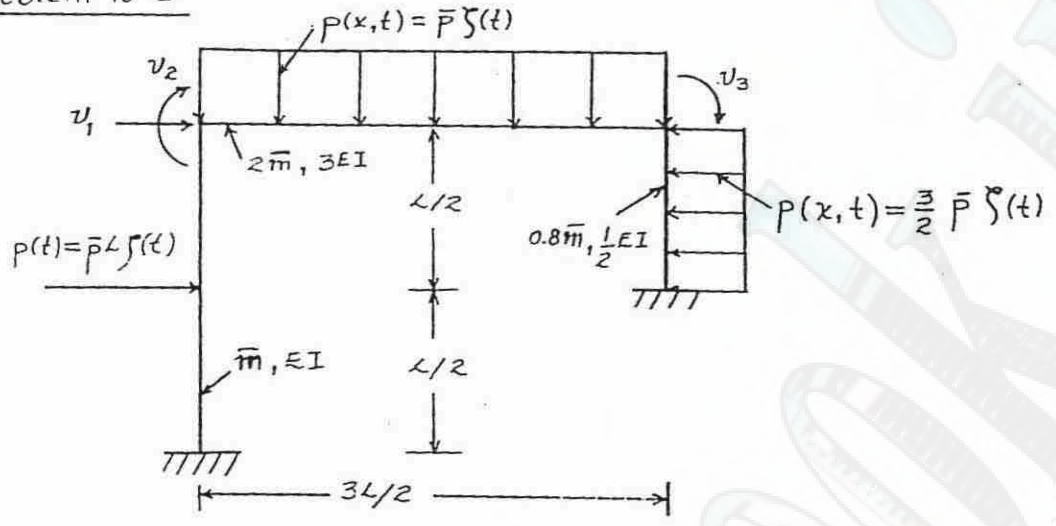
$$\therefore k_{G24} = \int_0^L N_0\left(2-\frac{x}{L}\right)\frac{6}{L}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]\left[3\left(\frac{x}{L}\right)^2 - 2\frac{x}{L}\right]dx$$

$$k_{G24} = \frac{6N_0}{L}\int_0^L\left[3\left(\frac{x}{L}\right)^5 - 11\left(\frac{x}{L}\right)^4 + 12\left(\frac{x}{L}\right)^3 - 4\left(\frac{x}{L}\right)^2\right]dx$$

$$k_{G24} = \frac{6N_0}{L}\left(\frac{1}{2} - \frac{11}{5} + 3 - \frac{4}{3}\right)L = -\frac{1}{5}N_0$$

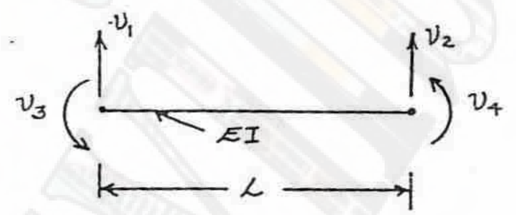
► $\boxed{k_{G24} = -\frac{1}{5}N_0}$

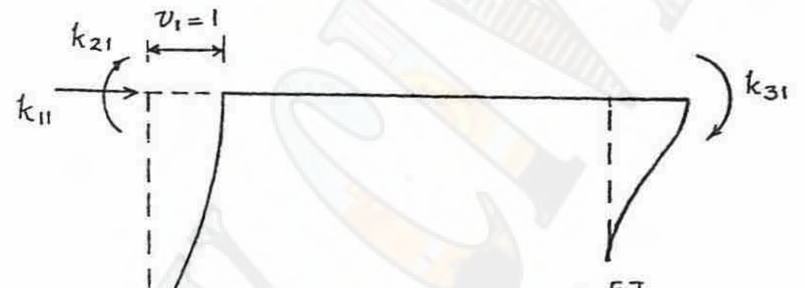
Problem 10-5



$$\text{Eq. (10-22):}\quad \begin{Bmatrix} f_{S1} \\ f_{S2} \\ f_{S3} \\ f_{S4} \end{Bmatrix} = \frac{2EI}{L^3}\begin{bmatrix} 6 & -6 & 3L & 3L \\ -6 & 6 & -3L & -3L \\ 3L & -3L & 2L^2 & L^2 \\ 3L & -3L & L^2 & 2L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

is for





$$k_{11} = \frac{2EI}{L^3}\,6 + \frac{2\,\frac{EI}{2}}{\left(\frac{L}{2}\right)^3}\,6 \doteq \frac{2EI}{L^3}\,30$$

$$k_{21} = -\frac{2EI}{L^3}\,3L$$

$$k_{31} = -\frac{2\,\frac{EI}{2}}{\left(\frac{L}{2}\right)^3}\,3\,\frac{L}{2} = -\frac{2EI}{L^3}\,6L$$

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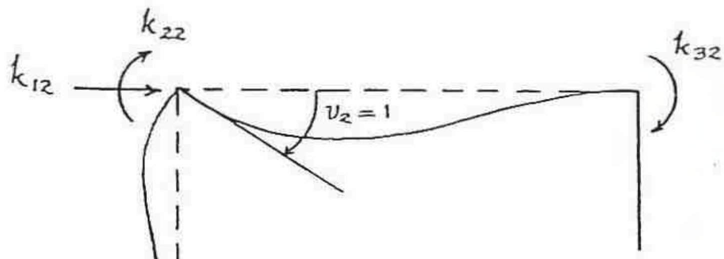
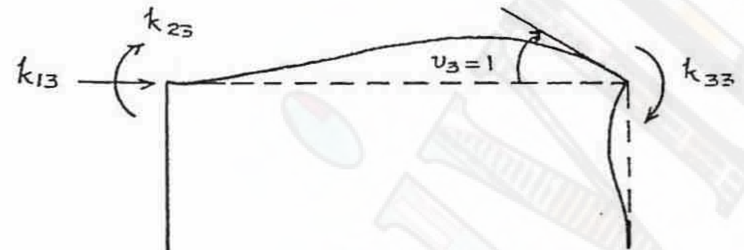
Problem 10-5 (con'd)



$$k_{12} = -\frac{2EI}{L^3}\,3L$$

$$k_{22} = \frac{2(3EI)}{\left(\frac{3L}{2}\right)^3}\,2\left(\frac{3L}{2}\right)^2 + \frac{2EI}{L^3}\,2L^2 = \frac{2EI}{L^3}\,6L^2$$

$$k_{32} = \frac{2(3EI)}{\left(\frac{3L}{2}\right)^3}\left(\frac{3L}{2}\right)^2 = \frac{2EI}{L^3}\,2L^2$$



$$k_{13} = -\frac{2\,\frac{EI}{2}}{\left(\frac{L}{2}\right)^3}\,3\,\frac{L}{2} = -\frac{2EI}{L^3}\,6L$$

$$k_{23} = \frac{2EI}{L^3}\,2L^2$$

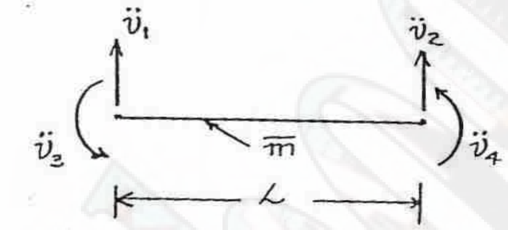
$$k_{33} = \frac{2(3EI)}{\left(\frac{3L}{2}\right)^3}\,2\left(\frac{3L}{2}\right)^2 + \frac{2\,\frac{EI}{2}}{\left(\frac{L}{2}\right)^3}\,2\left(\frac{L}{2}\right)^2 = \frac{2EI}{L^3}\,6L^2$$

$$\blacktriangleright\ [k] = \frac{2EI}{L^3}\begin{bmatrix} 30 & -3L & -6L \\ -3L & 6L^2 & 2L^2 \\ -6L & 2L^2 & 6L^2 \end{bmatrix}$$

Problem 10-6

$$Eq.\ (10\text{-}29):\quad \begin{Bmatrix} f_{I_1} \\ f_{I_2} \\ f_{I_3} \\ f_{I_4} \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 156 & 54 & 22L & -13L \\ 54 & 156 & 13L & -22L \\ 22L & 13L & 4L^2 & -3L^2 \\ -13L & -22L & -3L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \\ \ddot{v}_3 \\ \ddot{v}_4 \end{Bmatrix}$$

is for



Following the same procedure of P10-5 ,

for $\ddot{v}_1 = 1$:

$$m_{11} = \frac{\bar{m}L}{420}\,156 + \frac{0.8\bar{m}\,\frac{L}{2}}{420}\,156 + 2\bar{m}\,\frac{3L}{2} = \frac{\bar{m}L}{420}\,1478.4$$

$$m_{21} = -\frac{\bar{m}L}{420}\,22L$$

$$m_{31} = -\frac{0.8\bar{m}\,\frac{L}{2}}{420}\,22\,\frac{L}{2} = -\frac{\bar{m}L}{420}\,4.4L$$

for $\ddot{v}_2 = 1$:

$$m_{12} = -\frac{\bar{m}L}{420}\,22L$$

$$m_{22} = \frac{\bar{m}L}{420}\,4L^2 + \frac{2\bar{m}\,\frac{3L}{2}}{420}\,4\left(\frac{3L}{2}\right)^2 = \frac{\bar{m}L}{420}\,31L^2$$

$$m_{32} = -\frac{2\bar{m}\,\frac{3L}{2}}{420}\,3\left(\frac{3L}{2}\right)^2 = -\frac{\bar{m}L}{420}\,\frac{81}{4}L^2$$

for $\ddot{v}_3 = 1$:

$$m_{13} = -\frac{0.8\bar{m}\,\frac{L}{2}}{420}\,22\left(\frac{L}{2}\right) = -\frac{\bar{m}L}{420}\,4.4L$$

$$m_{23} = -\frac{\bar{m}L}{420}\,\frac{81}{4}L^2$$

$$m_{33} = \frac{2\bar{m}\,\frac{3L}{2}}{420}\,4\left(\frac{3L}{2}\right)^2 + \frac{0.8\bar{m}\,\frac{L}{2}}{420}\,4\left(\frac{L}{2}\right)^2 = \frac{\bar{m}L}{420}\,27.4L^2$$

(continued on following page)

Problem 10-6 (con'd)

$$[m] = \frac{\bar{m}L}{8400}\begin{bmatrix} 29568 & -440L & -88L \\ -440L & 620L^2 & -405L^2 \\ -88L & -405L^2 & 548L^2 \end{bmatrix}$$

Problem 10-7

Eq. (10-32): $p_i(t) = \int_0^L p(x,t)\,\psi_i(x)\,dx \quad , \quad i = 1,2,3,4$

where

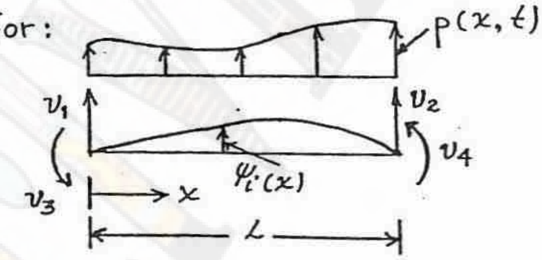
$$\psi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad \text{eq. (10-16a)}$$

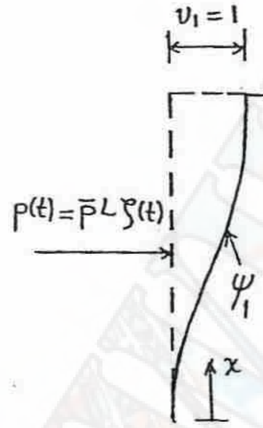
$$\psi_2(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \qquad \text{eq. (10-16b)}$$

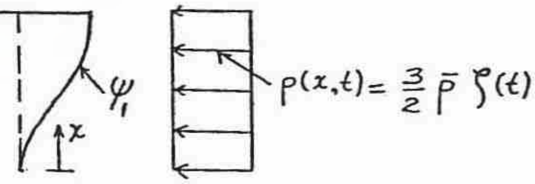
$$\psi_3(x) = x\left(1 - \frac{x}{L}\right)^2 \qquad \text{eq. (10-16c)}$$

$$\psi_4(x) = \frac{x^2}{L}\left(\frac{x}{L} - 1\right) \qquad \text{eq. (10-16d)}$$

for:







$$p_1(t) = p(t)\,\psi_1^L\left(x = \frac{L}{2}\right) - \int_0^L p(x,t)\,\psi_1^{L/2}(x)\,dx$$

$$p_1(t) = \bar{p}L\varsigma(t)\left[1 - 3\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^3\right] +$$

$$- \int_0^{L/2} \frac{3}{2}\bar{p}\,\varsigma(t)\left[1 - 3\left(\frac{x}{L/2}\right)^2 + 2\left(\frac{x}{L/2}\right)^3\right]dx$$

$$p_1(t) = \frac{1}{2}\bar{p}L\varsigma(t) - \frac{3}{2}\bar{p}\,\varsigma(t)\left(1 - 1 + \frac{1}{2}\right)\frac{L}{2} = \frac{1}{8}\bar{p}L\varsigma(t)$$

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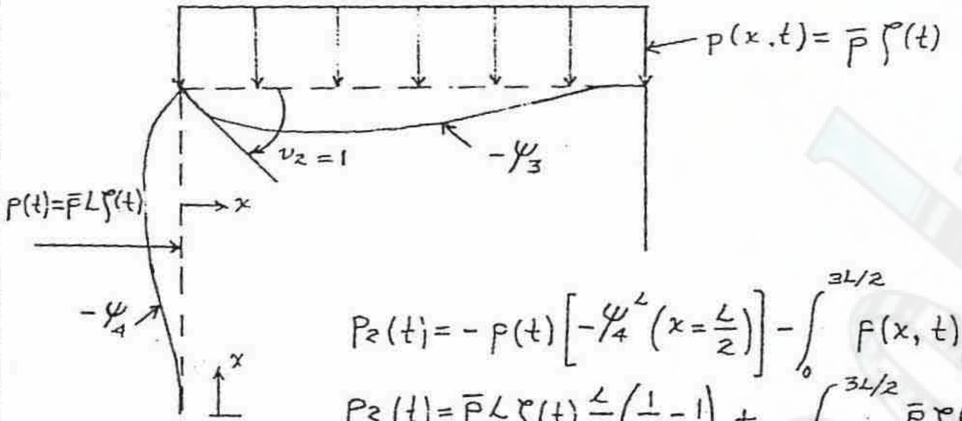
Problem 10-7 (con'd)



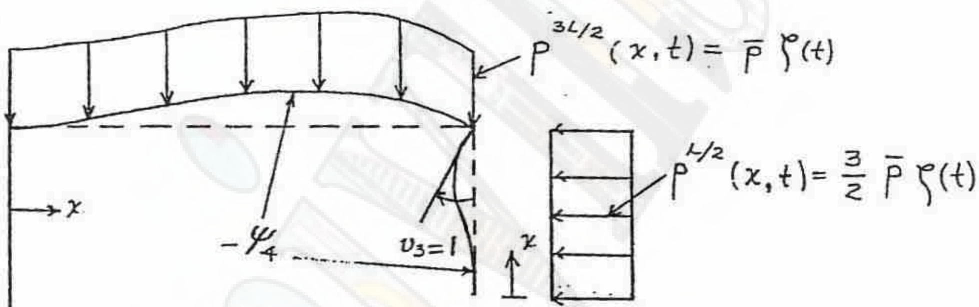
$$P_2(t) = -p(t)\left[-\psi_4^{L}\left(x=\frac{L}{2}\right)\right] - \int_0^{3L/2} p(x,t)\left[-\psi_3^{3L/2}(x)\right]dx$$

$$P_2(t) = \bar{p}L\zeta(t)\frac{L}{4}\left(\frac{1}{2}-1\right) + \int_0^{3L/2} \bar{p}\zeta(t)\,x\left(1-\frac{x}{3L/2}\right)^2 dx$$

$$P_2(t) = -\frac{1}{8}\bar{p}L^2\zeta(t) + \bar{p}\zeta(t)\frac{3L}{2}\int_0^{3L/2}\left[\frac{x}{3L/2} - 2\left(\frac{x}{3L/2}\right)^2 + \left(\frac{x}{3L/2}\right)^3\right]dx$$

$$P_2(t) = -\frac{1}{8}\bar{p}L^2\zeta(t) + \bar{p}\zeta(t)\frac{3L}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\frac{3L}{2}$$

$$P_2(t) = -\frac{1}{8}\bar{p}L^2\zeta(t) + \frac{\overset{3}{\cancel{9}}}{4}\bar{p}L^2\zeta(t)\frac{1}{\underset{4}{\cancel{12}}} = \frac{1}{16}\bar{p}L^2\zeta(t)$$



$$P_3(t) = -\int_0^{3L/2} p^{3L/2}(x,t)\left[-\psi_4^{3L/2}(x)\right]dx + \int_0^{L/2} p^{L/2}(x,t)\left[-\psi_4^{L/2}(x)\right]dx$$

$$P_3(t) = \int_0^{3L/2}\bar{p}\zeta(t)\frac{x^2}{3L/2}\left(\frac{x}{3L/2}-1\right)dx - \int_0^{L/2}\frac{3}{2}\bar{p}\zeta(t)\frac{x^2}{L/2}\left(\frac{x}{L/2}-1\right)dx$$

$$P_3(t) = \bar{p}\zeta(t)\frac{3L}{2}\int_0^{3L/2}\left[\left(\frac{x}{3L/2}\right)^3 - \left(\frac{x}{3L/2}\right)^2\right]dx - \frac{3}{2}\bar{p}\zeta(t)\frac{L}{2}\int_0^{L/2}\left[\left(\frac{x}{L/2}\right)^3 - \left(\frac{x}{L/2}\right)^2\right]dx$$

$$P_3(t) = \frac{3}{2}\bar{p}L\zeta(t)\left(\frac{1}{4}-\frac{1}{3}\right)\frac{3L}{2} - \frac{3}{4}\bar{p}L\zeta(t)\left(\frac{1}{4}-\frac{1}{3}\right)\frac{L}{2}$$

$$P_3(t) = -\frac{\overset{3}{\cancel{9}}}{4}\bar{p}L^2\zeta(t)\frac{1}{\underset{4}{\cancel{12}}} + \frac{\overset{1}{\cancel{3}}}{8}\bar{p}L^2\zeta(t)\frac{1}{\underset{4}{\cancel{12}}} = -\frac{5}{32}\bar{p}L^2\zeta(t)$$

$$\blacktriangleright \quad \{F(t)\} = \frac{1}{32}\bar{p}L\zeta(t)\begin{Bmatrix} 4 \\ 2L \\ -5L \end{Bmatrix}$$

Problem 10-8

(a)

$$[k] = \frac{EI}{L^3}\left[\begin{array}{c|cc} 20 & -10L & -5L \\ \hline -10L & 15L^2 & -8L^2 \\ -5L & -8L^2 & 12L^2 \end{array}\right] \equiv \begin{bmatrix} [k_{tt}] & [k_{t\theta}] \\ [k_{\theta t}] & [k_{\theta\theta}] \end{bmatrix}$$

Eq. (10-47) : $[k_t] = [k_{tt}] - [k_{t\theta}][k_{\theta\theta}]^{-1}[k_{\theta t}]$

$$[k_t] = \frac{EI}{L^3}20 - \frac{EI}{L^3}\begin{bmatrix} -10L & -5L \end{bmatrix}\left[\frac{EI}{L^3}\begin{bmatrix} 15L^2 & -8L^2 \\ -8L^2 & 12L^2 \end{bmatrix}\right]^{-1}\frac{EI}{L^3}\begin{bmatrix} -10L \\ -5L \end{bmatrix}$$

$$[k_t] = \frac{EI}{L^3}20 - \frac{EI}{L^3}(-5L)\begin{bmatrix} 2 & 1 \end{bmatrix}\frac{L^3}{EI}\cdot\frac{1}{L^2}\begin{bmatrix} 15 & -8 \\ -8 & 12 \end{bmatrix}^{-1}\frac{EI}{L^3}(-5L)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[k_t] = \frac{EI}{L^3}20 - \frac{EI}{L^3}25\begin{bmatrix} 2 & 1 \end{bmatrix}\frac{1}{116}\begin{bmatrix} 12 & 8 \\ 8 & 15 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[k_t] = \frac{EI}{L^2}20 - \frac{EI}{L^3}\frac{25}{116}\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 32 \\ 31 \end{bmatrix} = \frac{EI}{L^3}20 - \frac{EI}{L^3}\frac{25}{116}(95) = -\frac{55}{116}\frac{EI}{L^3}$$

$$\blacktriangleright\ \boxed{k_t = -\frac{55}{116}\frac{EI}{L^3}}$$

Note: This is not a stiffness matrix for a real structure

(b) Eq. of motion for undamped free vibrations:

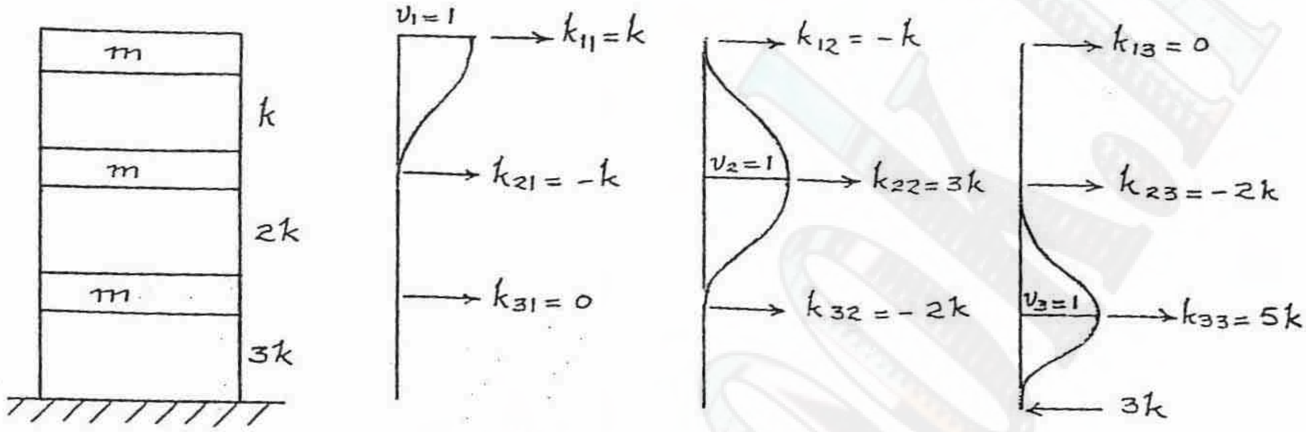
$$m\ddot{v}(t) + \cancelto{0}{c\dot{v}(t)} + k\,v(t) = 0$$

$$30\bar{m}L\,\ddot{v}(t) - \frac{55}{116}\frac{EI}{L^3}v(t) = 0$$

$$\blacktriangleright\ \boxed{6\bar{m}L\,\ddot{v}(t) - \frac{11}{116}\frac{EI}{L^3}v(t) = 0}$$

Problem 11-1

Direct evaluation of the stiffness and mass matrices:



$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv m\,I \,, \quad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

(a) From eq. (11-6):

$$\|[k] - \omega^2[m]\| = \det\left(k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & -2 & 5 \end{bmatrix} - \omega^2 m I\right) = k\det\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & -2 & 5 \end{bmatrix} - \omega^2\frac{m}{k} I\right) = 0$$

defining $\lambda \equiv \omega^2 \dfrac{m}{k}$

$$\det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} = -(\lambda^3 - 9\lambda^2 - 18\lambda - 6) = -(\lambda - 0.416)(\lambda - 2.29)(\lambda - 6.29) = 0$$

$$\therefore\ \omega_1 = 0.645\sqrt{\frac{k}{m}} = 9.12\ ^1/_{sec}\,, \quad \omega_2 = 1.51\sqrt{\frac{k}{m}} = 21.4\ ^1/_{sec}\,,$$

$$\omega_3 = 2.51\sqrt{\frac{k}{m}} = 35.5\ ^1/_{sec}$$

$$\blacktriangleright\ \boxed{\{\omega\}^T = \langle 9.12,\ 21.4,\ 35.5 \rangle\ ^1/_{sec}}$$

(continued on following page)

Problem 11-1 (con'd.)

(b) From eq. (11-4): $\left[[k]-\omega^2[m]\right]\{\hat{v}\} = k\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix}\begin{Bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{Bmatrix} = \{0\}$

for $\lambda_1 = 0.416$:

$$\begin{bmatrix} 0.584 & -1 & 0 \\ -1 & 2.584 & -2 \\ 0 & -2 & 4.584 \end{bmatrix}\begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \\ \hat{v}_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{11}=1,\ \hat{v}_{21}=0.584,\ \hat{v}_{31}=0.255$$

$\lambda_2 = 2.29$:

$$\begin{bmatrix} -1.29 & -1 & 0 \\ -1 & 0.71 & -2 \\ 0 & -2 & 2.71 \end{bmatrix}\begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \\ \hat{v}_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{12}=1,\ \hat{v}_{22}=-1.29,\ \hat{v}_{32}=-0.952$$

$\lambda_3 = 6.29$:

$$\begin{bmatrix} -5.29 & -1 & 0 \\ -1 & -3.29 & -2 \\ 0 & -2 & -1.29 \end{bmatrix}\begin{Bmatrix} \hat{v}_{13} \\ \hat{v}_{23} \\ \hat{v}_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{13}=1,\ \hat{v}_{23}=-5.29,\ \hat{v}_{33}=8.20$$

$$\blacktriangleright\ [\hat{v}] = \begin{bmatrix} 1 & 1 & 1 \\ 0.584 & -1.29 & -5.29 \\ 0.255 & -0.952 & 8.20 \end{bmatrix}$$

(c) To satisfy the orthogonality conditions, the mode shapes have to verify eqs. (11-36) and (11-37):

$$\{\hat{v}_m\}^T[m]\{\hat{v}_n\} = 0$$

$$\{\hat{v}_m\}^T[k]\{\hat{v}_n\} = 0 \quad , \ \omega_m \neq \omega_n$$

Since $\{A\}^T[C]\{B\} = \{B\}^T[C]^T\{A\}$ and $[C]=[C]^T$, for those two cases, it is not necessary to make 6 matrix multiplications.

– Orthogonality with respect to $[m]$:

$$O^m_{mn} \equiv \{\hat{v}_m\}^T[m]\{\hat{v}_n\} = \{\hat{v}_m\}^T mI\{\hat{v}_n\} = m\{\hat{v}_m\}^T\{\hat{v}_n\}$$

(continued on following page)

Problem 11-1 (con'd)

$$O_{12}^{m} = m < 1, 0.584, 0.255 > \begin{Bmatrix} 1 \\ -1.29 \\ -0.952 \end{Bmatrix} = 0.00388\, m$$

$$O_{13}^{m} = m < 1, 0.584, 0.255 > \begin{Bmatrix} 1 \\ -5.29 \\ 8.20 \end{Bmatrix} = 0.00164\, m$$

$$O_{23}^{m} = m < 1, -1.29, -0.952 > \begin{Bmatrix} 1 \\ -5.29 \\ 8.20 \end{Bmatrix} = 0.0177\, m$$

<u>ok!</u>

- Orthogonality with respect to $[k]$:

$$O_{mn}^{k} \equiv \{\hat{v}_m\}^T [k] \{\hat{v}_n\}$$

$$O_{12}^{k} = < 1, 0.584, 0.255 > k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ -1.29 \\ -0.952 \end{Bmatrix} = 0.00196\, k$$

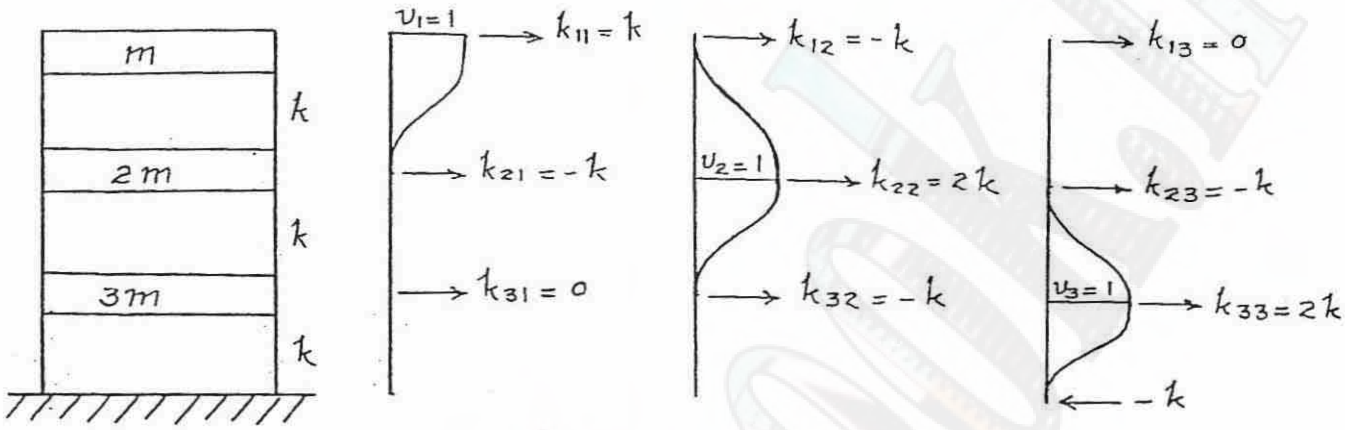
$$O_{13}^{k} = < 1, 0.584, 0.255 > k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ -5.29 \\ 8.20 \end{Bmatrix} = 0.0132\, k$$

$$O_{23}^{k} = < 1, -1.29, -0.952 > k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ -5.29 \\ 8.20 \end{Bmatrix} = 0.104\, k$$

<u>ok!</u>

Problem 11-2

Following P11-1,



$$[m]=m\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\quad,\quad [k]=k\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$

(a)

$$\|[k]-\omega^2[m]\| = \det\left(k\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}-\omega^2 m\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\right)=k\det\begin{bmatrix}1-\lambda&-1&0\\-1&2-2\lambda&-1\\0&-1&2-3\lambda\end{bmatrix}=0$$

where $\lambda \equiv \omega^2 \dfrac{m}{k}$

$$\det\begin{bmatrix}1-\lambda&-1&0\\-1&2(1-\lambda)&-1\\0&-1&2-3\lambda\end{bmatrix}=-(6\lambda^3-16\lambda^2+10\lambda-1)=-6(\lambda-0.1231)(\lambda-0.7580)(\lambda-1.7855)=0$$

$$\therefore\ \omega_1 = 0.3509\sqrt{\frac{k}{m}} = 9.92\ ^1/_{sec}\ ;\ \omega_2 = 0.8706\sqrt{\frac{k}{m}} = 24.6\ ^1/_{sec}\ ,$$

$$\omega_3 = 1.3362\sqrt{\frac{k}{m}} = 37.8\ ^1/_{sec}$$

$$\blacktriangleright\ \underline{\underline{\{\omega\}^T = \langle 9.92\,,\ 24.6\,,\ 37.8\rangle\ ^1/_{sec}}}$$

(b)

$$\left[[k]-\omega^2[m]\right]\{\hat{v}\} = k\begin{bmatrix}1-\lambda&-1&0\\-1&2(1-\lambda)&-1\\0&-1&2-3\lambda\end{bmatrix}\begin{Bmatrix}\hat{v}_1\\\hat{v}_2\\\hat{v}_3\end{Bmatrix}=\{0\}$$

(continued on following page)

Problem 11-2 (con'd)

for $\lambda_1 = 0.1231$:

$$\begin{bmatrix} 0.8769 & -1 & 0 \\ -1 & 1.7538 & -1 \\ 0 & -1 & 1.6307 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \\ \hat{v}_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ for } \hat{v}_{11} = 1,\ \hat{v}_{21} = 0.8769,\ \hat{v}_{31} = 0.5377$$

$\lambda_2 = 0.7580$:

$$\begin{bmatrix} 0.2420 & -1 & 0 \\ -1 & 0.4840 & -1 \\ 0 & -1 & -0.2740 \end{bmatrix} \begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \\ \hat{v}_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ for } \hat{v}_{12} = 1,\ \hat{v}_{22} = 0.2420,\ \hat{v}_{32} = -0.8832$$

$\lambda_3 = 1.7855$:

$$\begin{bmatrix} -0.7855 & -1 & 0 \\ -1 & -1.5710 & -1 \\ 0 & -1 & -3.3565 \end{bmatrix} \begin{Bmatrix} \hat{v}_{13} \\ \hat{v}_{23} \\ \hat{v}_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ for } \hat{v}_{13} = 1,\ \hat{v}_{23} = -0.7855,\ \hat{v}_{33} = 0.2340$$

$$\blacktriangleright\ [\hat{v}] = \begin{bmatrix} 1 & 1 & 1 \\ 0.877 & 0.242 & -0.786 \\ 0.538 & -0.883 & 0.234 \end{bmatrix}$$

(c) Orthogonality with respect to $[m]$: $O^m_{mn} \equiv \{v_m\}^T [m] \{v_n\}$

$$O^m_{12} = \langle 1, 0.877, 0.538 \rangle m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.242 \\ -0.883 \end{Bmatrix} = -0.000694\, m$$

$$O^m_{13} = \langle 1, 0.877, 0.538 \rangle m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.786 \\ 0.234 \end{Bmatrix} = -0.000968\, m$$

$$O^m_{23} = \langle 1, 0.242, -0.883 \rangle m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.786 \\ 0.234 \end{Bmatrix} = -0.000294\, m$$

ok!

(continued on following page)

Problem 11-2 (con'd)

Orthogonality with respect to $[k]$: $O_{mn}^{k} \equiv \{v_m\}^T [k] \{v_n\}$

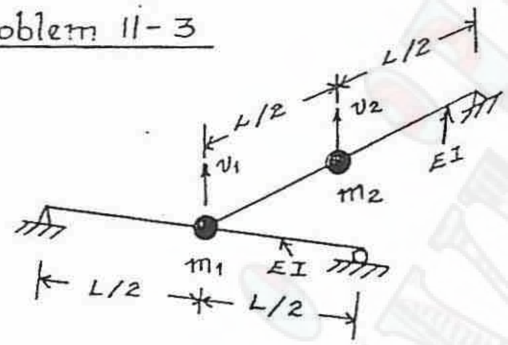
$$O_{12}^{k} = \langle 1, 0.877, 0.538 \rangle k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.242 \\ -0.883 \end{Bmatrix} = 0.000445\,k$$

$$O_{13}^{k} = \langle 1, 0.877, 0.538 \rangle k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.786 \\ 0.234 \end{Bmatrix} = 0.000210\,k$$

$$O_{23}^{k} = \langle 1, 0.242, -0.883 \rangle k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.786 \\ 0.234 \end{Bmatrix} = 0.000334\,k$$

OK!

Problem 11-3



L, length, $= 20$ ft

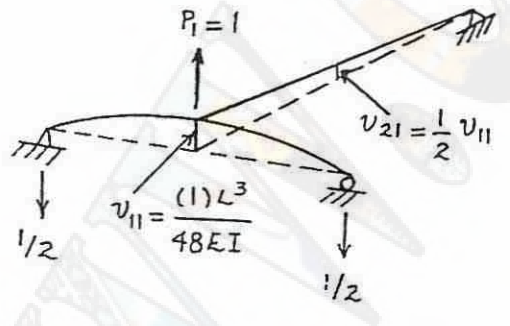
$EI = 6 \times 10^4$ kips-ft^2

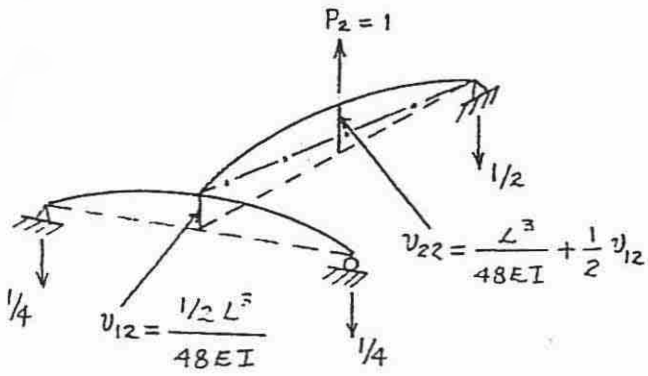
$$m_1 = \frac{1}{2}\bar{m}L + \frac{1}{4}\bar{m}L = \frac{3}{4}\bar{m}L$$

$$m_2 = \frac{1}{2}\bar{m}L + \frac{W}{g} = \frac{1}{4}\bar{m}L\left(2 + \frac{2W}{\bar{m}Lg}\right)$$

$\bar{m}g = 0.3$ kips/ft , $W = 3$ kips

Direct evaluation of the flexibility and mass matrices:





$$\therefore\ [m] = \frac{\bar{m}L}{4}\begin{bmatrix} 3 & 0 \\ 0 & 2 + \dfrac{2W}{\bar{m}Lg} \end{bmatrix} , \quad [\tilde{f}] = \frac{L^3}{192\,EI}\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

(continued on following page)

Problem 11-3 (con'd)

Eq. (11-18): $\left\| \frac{1}{\omega^2} I - [\tilde{f}][m] \right\| = \det \left(\frac{1}{\omega^2} I - \frac{\bar{m} L^4}{768 EI} \begin{bmatrix} 12 & 2\left(2 + \frac{2W}{\bar{m}Lg}\right) \\ 6 & 5\left(2 + \frac{2W}{\bar{m}Lg}\right) \end{bmatrix} \right) = 0$

defining $\lambda = \frac{768 EI}{\omega^2 \bar{m} L^4}$

$$\frac{mL^4}{768EI} \det \begin{bmatrix} 12-\lambda & 2\left(2 + \frac{2W}{\bar{m}Lg}\right) \\ 6 & 5\left(2 + \frac{2W}{\bar{m}Lg}\right) - \lambda \end{bmatrix} = 0 \rightarrow \lambda = \frac{12+5a}{2} \pm \sqrt{\left(\frac{12+5a}{2}\right)^2 - 48a}$$

where

$$a = 2 + \frac{2W}{\bar{m}Lg} = 2 + \frac{(2)\ 3\text{ kips}}{(0.3\text{ kips/ft})(10\text{ft})} = 4$$

$$\therefore \lambda = 16 \pm 8 = 8,\ 24$$

Since $\omega = \sqrt{\frac{768 EI}{\lambda \bar{m} L^4}} = \lambda^{-1/2} \sqrt{\frac{768\ (6 \times 10^4 \text{ kips}\cdot\text{ft}^2)}{(0.3\text{ kips/ft})(32.2\text{ ft/sec}^2)^{-1}(20^4\text{ft}^4)}} = 175.8\,\lambda^{-1/2}\text{sec}^{-1}$

$$\omega_1 = 35.89\ \text{1/sec}\ ,\quad \omega_2 = 62.16\ \text{1/sec}$$

Eq. (11-17): $\left[\frac{1}{\omega^2} I - [\tilde{f}][m]\right]\{\hat{v}\} = \{0\} \rightarrow \frac{\bar{m}L^4}{768EI} \begin{bmatrix} 12-\lambda & 8 \\ 6 & 15-\lambda \end{bmatrix} \begin{Bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{Bmatrix} = \{0\}$

$\therefore$ for $\lambda_1 = 24$:

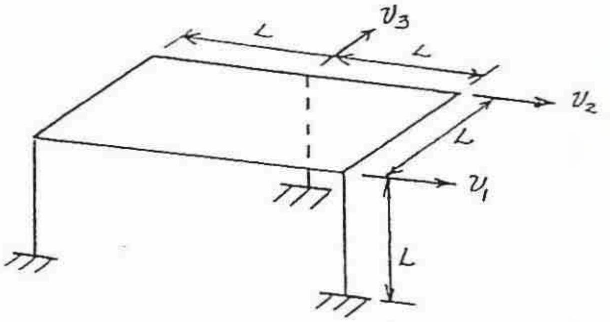
$$\begin{bmatrix} -12 & 8 \\ 6 & -9 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{11} = 1,\ \hat{v}_{21} = 2/3$$

$\lambda_2 = 8$:

$$\begin{bmatrix} 4 & 8 \\ 6 & 7 \end{bmatrix} \begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{12} = 1,\ \hat{v}_{22} = -6/7$$

$$\blacktriangleright\ \{\omega\} = \begin{Bmatrix} 35.9 \\ 62.2 \end{Bmatrix}\ \text{1/sec}\ ,\quad [\hat{v}] = \begin{bmatrix} 1 & 1 \\ 0.667 & -0.857 \end{bmatrix}$$

Problem 11-4



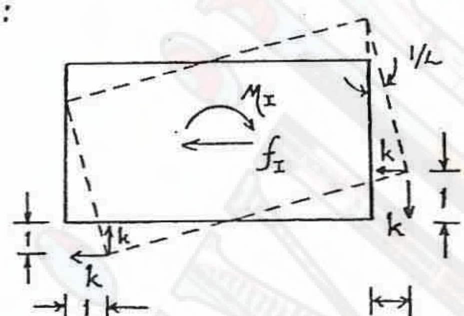
Rigid slab : total mass, m

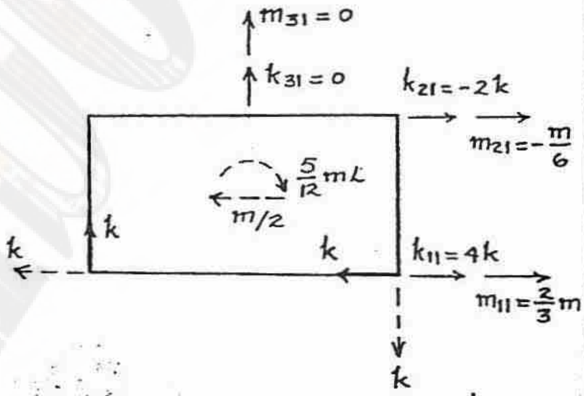
Weightless columns : flexural stiffness, EI

From Fig. E26-6, stiffness of each column, $k = \dfrac{12EI}{L^3}$ (any direction)

(a) Evaluation of stiffness and mass coefficients (neglecting torsional stiffness). Direct generation.

– $v_1 = 1$:
($\ddot{v}_1 = 1$)





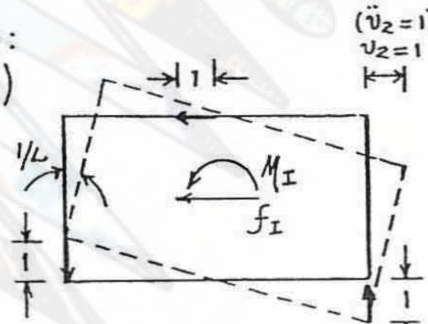
$$f_I = m\,\frac{1}{2}$$

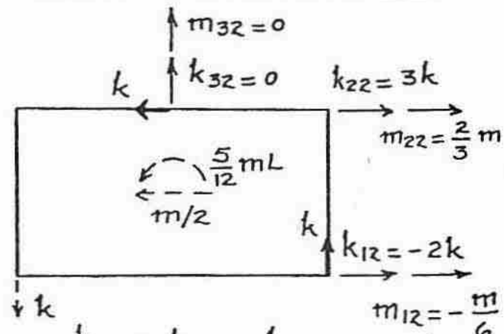
$$M_I = m\,\frac{4L^2 + L^2}{12}\cdot\frac{1}{L} = \frac{5}{12}\,mL$$

$$k_{11} + k_{21} = 2k$$
$$k_{21}L = -k\,2L$$

$$m_{11} + m_{21} = \frac{m}{2}$$
$$m_{11}\frac{L}{2} - m_{21}\frac{L}{2} = \frac{5}{12}\,mL$$

– $v_2 = 1$:
($\ddot{v}_2 = 1$)





$$f_I = m\,\frac{1}{2}$$

$$M_I = m\,\frac{4L^2 + L^2}{12}\cdot\frac{1}{L} = \frac{5}{12}\,mL$$

$$k_{12} + k_{22} = k$$
$$k_{22}L = k\,2L + kL$$

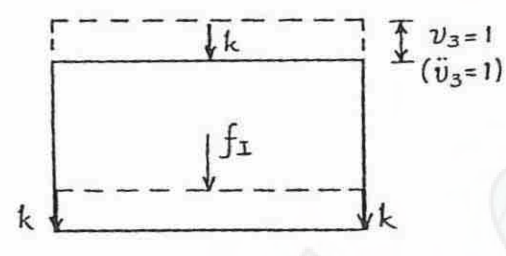
(continued on following page)

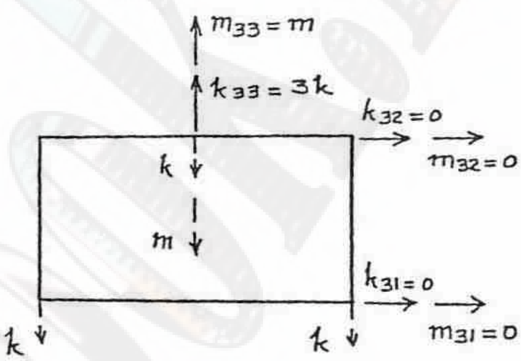
Problem 11-4 (con'd)

$$m_{12} + m_{22} = \frac{m}{2}$$

$$m_{12}\frac{L}{2} + m_{22}\frac{L}{2} = -\frac{5}{12}mL$$

– $v_3 = 1$:
($\ddot{v}_3 = 1$)





$f_I = m(1)$

$$[k] = \frac{12EI}{L^3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [m] = m\begin{bmatrix} 2/3 & -1/6 & 0 \\ -1/6 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Eq. (11-6) : $\|[k] - \omega^2[m]\| = 0$

$$\|[k] - \omega^2[m]\| = \det\left(\frac{12EI}{L^3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \omega^2\frac{m}{6}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

$$\|[k] - \omega^2[m]\| = \frac{12EI}{L^3}\det\left(\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \omega^2\frac{mL^3}{72EI}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

$$\|[k] - \omega^2[m]\| = \frac{12EI}{L^3}\det\begin{bmatrix} 4-4\lambda & -2+\lambda & 0 \\ -2+\lambda & 3-4\lambda & 0 \\ 0 & 0 & 3-6\lambda \end{bmatrix} = \frac{12EI}{L^3}60(\lambda-0.473)(\lambda-0.5)(\lambda-1.127) = 0$$

where $\lambda \equiv \omega^2\dfrac{mL^3}{72EI}$

$$\therefore\ \omega_1 = 5.84\sqrt{\frac{EI}{mL^3}},\quad \omega_2 = 6.00\sqrt{\frac{EI}{mL^3}},\quad \omega_3 = 9.01\sqrt{\frac{EI}{mL^3}}$$

Eq. (11-4): $[[k] - \omega^2[m]]\{\hat{v}\} = \{0\} \longrightarrow \dfrac{12EI}{L^3}\begin{bmatrix} 4-4\lambda & -2+\lambda & 0 \\ -2+\lambda & 3-4\lambda & 0 \\ 0 & 0 & 3-6\lambda \end{bmatrix}\begin{Bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{Bmatrix} = \{0\}$

(continued on following page)

Problem 11-4 (con'd)

For $\lambda_1 = 0.473$:

$$\begin{bmatrix} 2.108 & -1.527 & 0 \\ -1.527 & 1.108 & 0 \\ 0 & 0 & 0.162 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \\ \hat{v}_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{11} = 1 \text{ , } \hat{v}_{21} = 1.379, \quad \hat{v}_{31} = 0$$

For $\lambda_2 = 0.5$:

$$\begin{bmatrix} 2.0 & -1.5 & 0 \\ -1.5 & 1.0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \\ \hat{v}_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \hat{v}_{12} = \hat{v}_{22} = 0 \quad \text{let } \hat{v}_{32} = 1$$

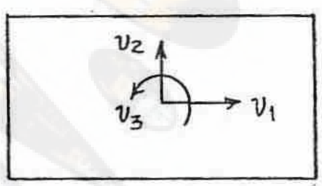
For $\lambda_3 = 1.127$:

$$\begin{bmatrix} -0.508 & -0.873 & 0 \\ -0.873 & -1.508 & 0 \\ 0 & 0 & -3.762 \end{bmatrix} \begin{Bmatrix} \hat{v}_{13} \\ \hat{v}_{23} \\ \hat{v}_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{for } \hat{v}_{13} = 1, \hat{v}_{23} = -0.580, \quad \hat{v}_{33} = 0$$

$$\blacktriangleright \{\omega\} = \begin{Bmatrix} 5.84 \\ 6 \\ 9.01 \end{Bmatrix} \sqrt{\frac{EI}{mL^3}} \text{ , } [\hat{v}] = \begin{bmatrix} 1 & 0 & 1 \\ 1.379 & 0 & -0.580 \\ 0 & 1 & 0 \end{bmatrix}$$

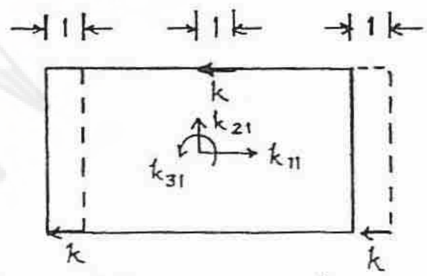
Problem 11-5

Following P11-4, for the degrees of freedom defined as



(a)

$- \; v_1 = 1$:
$(\ddot{v}_1 = 1)$



$k_{11} = 3k \qquad m_{11} = m$

$k_{21} = 0 \qquad m_{21} = m_{31} = 0$

$k_{31} = -k\frac{L}{2}$

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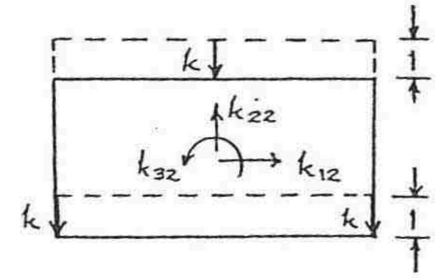
Problem 11-5 (con'd)

– $v_2 = 1$:
($\ddot{v}_2 = 1$)

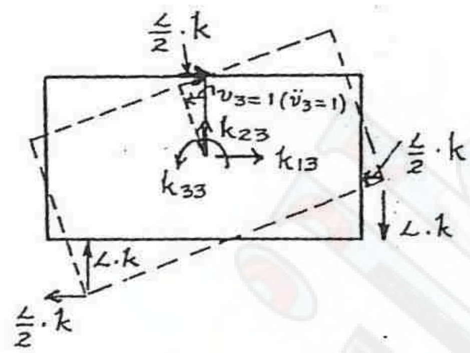


$k_{12} = 0 \qquad m_{12} = 0$

$k_{22} = 3k \qquad m_{22} = m$

$k_{32} = 0 \qquad m_{32} = 0$

– $v_3 = 1$:
($\ddot{v}_3 = 1$)



$k_{13} = \frac{L}{2} k \qquad m_{13} = m_{23} = 0$

$k_{23} = 0 \qquad m_{33} = m \frac{L^2 + 4L^2}{12} = \frac{5}{12} mL^2$

$k_{33} = \frac{L^2}{4} k$

$$\blacktriangleright [k] = \frac{3EI}{L^3} \begin{bmatrix} 12 & 0 & 2L \\ 0 & 12 & 0 \\ 2L & 0 & L^2 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{12} L^2 \end{bmatrix}$$

(b) $$\| [k] - \omega^2 [m] \| = \frac{12EI}{L^3} \det \left\{ \begin{bmatrix} 3 & 0 & L/2 \\ 0 & 3 & 0 \\ L/2 & 0 & (L/2)^2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{12} L^2 \end{bmatrix} \right\}$$

$$\| [k] - \omega^2 [m] \| = \frac{12EI}{L^3} \det \begin{bmatrix} 3-\lambda & 0 & L/2 \\ 0 & 3-\lambda & 0 \\ L/2 & 0 & (1 - \frac{5}{3}\lambda)(L/2)^2 \end{bmatrix}$$

$$\| [k] - \omega^2 [m] \| = \frac{12EI}{L^3} (3-\lambda) \left(\frac{L}{2}\right)^2 \frac{5}{3} (\lambda - 0.3717)(\lambda - 3.228) = 0 ,$$

where $\lambda = \omega^2 \frac{mL^3}{12EI}$

(continued on following page)

$$\therefore \; \omega_1 = 0.473\sqrt{\frac{EI}{mL^3}}, \quad \omega_2 = 0.167\sqrt{\frac{EI}{mL^3}}, \quad \omega_3 = 0.161\sqrt{\frac{EI}{mL^3}}$$

Since
$$\begin{bmatrix} 3-\lambda & 0 & \frac{L}{2} \\ 0 & 3-\lambda & 0 \\ \frac{L}{2} & 0 & \left(1-\frac{5}{3}\lambda\right)\left(\frac{L}{2}\right)^2 \end{bmatrix} \begin{Bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{Bmatrix} = \{0\},$$

for $\lambda_1 = 0.3717$:

$$\begin{bmatrix} 2.6283 & 0 & L/2 \\ 0 & 2.6283 & 0 \\ L/2 & 0 & 0.3805\,(L/2)^2 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \\ \hat{v}_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \hat{v}_{21} = 0$$

for $\hat{v}_{11} = 1$, $\hat{v}_{31} = -5.257/L$

$\lambda_2 = 3$:

$$\begin{bmatrix} 0 & 0 & L/2 \\ 0 & 0 & 0 \\ L/2 & 0 & -L^2 \end{bmatrix} \begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \\ \hat{v}_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \hat{v}_{12} = \hat{v}_{32} = 0$$

let $\hat{v}_{22} = 1$

$\lambda_3 = 3.228$:

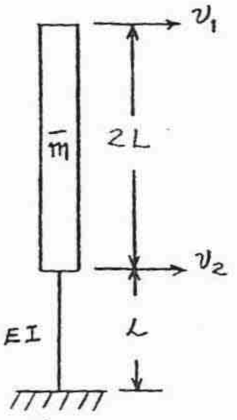
$$\begin{bmatrix} -0.228 & 0 & L/2 \\ 0 & -0.228 & 0 \\ L/2 & 0 & -4.38\,(L/2)^2 \end{bmatrix} \begin{Bmatrix} \hat{v}_{13} \\ \hat{v}_{23} \\ \hat{v}_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \hat{v}_{23} = 0$$

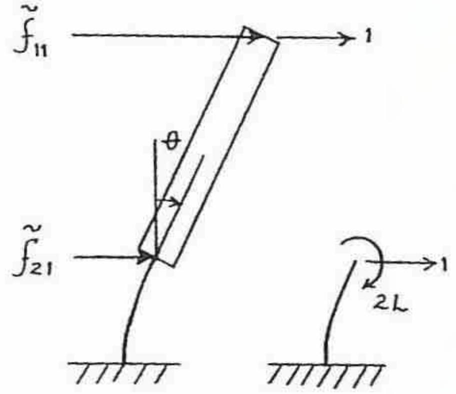
for $\hat{v}_{13} = 1$, $\hat{v}_{33} = 0.456/L$

$$\blacktriangleright \; \{\omega\} = \begin{Bmatrix} 0.453 \\ 0.167 \\ 0.161 \end{Bmatrix} \sqrt{\frac{EI}{mL^3}}, \quad [\hat{v}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{5.26}{L} & 0 & \frac{0.456}{L} \end{bmatrix}$$

Problem 11-6

(a)



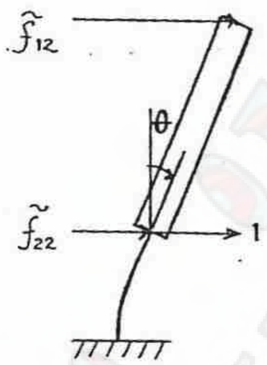


$$\tilde{f}_{21} = \frac{L^3}{3EI} + \frac{L^3}{EI} = \frac{4L^3}{3EI}$$

$$\theta = \frac{L^2}{2EI} + \frac{2L^2}{EI} = \frac{5L^2}{2EI}$$

$$\tilde{f}_{11} = \tilde{f}_{21} + (2L)\sin\theta$$

$$\tilde{f}_{11} \approx \tilde{f}_{21} + (2L)\,\theta = \frac{19L^3}{3EI}$$

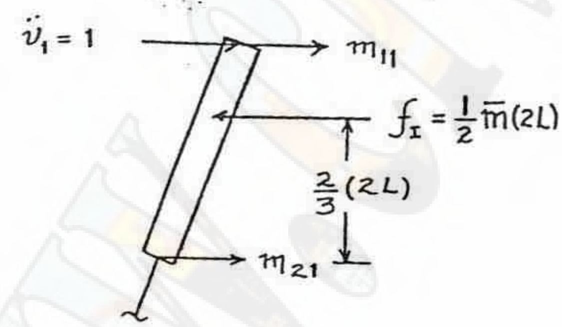


$$\tilde{f}_{22} = \frac{L^3}{3EI}$$

$$\theta = \frac{L^2}{2EI}$$

$$\tilde{f}_{12} \approx \tilde{f}_{22} + (2L)\theta = \frac{4L^3}{3EI}$$

$$k \sim \begin{bmatrix} 1 & -4 \\ -4 & 19 \end{bmatrix} \qquad \underset{\sim}{\tilde{f}} = \frac{L^3}{3EI}\begin{bmatrix} 19 & 4 \\ 4 & 1 \end{bmatrix}$$

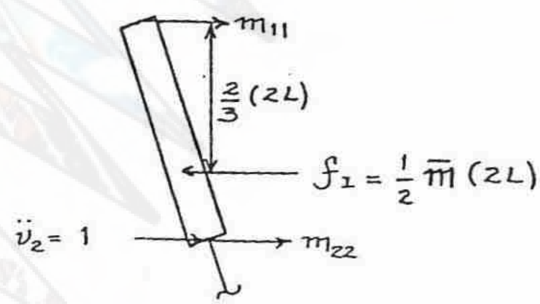


$$m_{11}(2L) - f_I\,\frac{2}{3}(2L) = 0$$

$$m_{11} = \frac{2}{3}\,\bar{m}L$$

$$m_{21}(2L) - f_I\,\frac{1}{3}(2L) = 0$$

$$m_{21} = \frac{1}{3}\,\bar{m}L$$



$$\underset{\sim}{m} = \frac{\bar{m}L}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(continued on following page)

Problem 11-6 (con'd)

(b)

$$\det\left[\frac{1}{\omega^2}\underline{I} - \underline{\tilde{f}}\,\underline{m}\right] = 0$$

let $\lambda = \dfrac{9EI}{\omega^2 \bar{m} L^4}$, $\det\left[\dfrac{1}{\omega^2}\underline{I} - \underline{\tilde{f}}\,\underline{m}\right] = \dfrac{\bar{m}L^4}{9EI}\det\begin{bmatrix} 42-\lambda & 27 \\ 9 & 6-\lambda \end{bmatrix}$

$$(42-\lambda)(6-\lambda) - 243 = \lambda^2 - 48\lambda + 9 = 0$$

$$\lambda_{1,2} = 47.818,\ 0.188238$$

$$\omega_{1,2} = 0.434,\ 6.91\sqrt{\frac{EI}{\bar{m}L^4}}$$

$$\left[\frac{1}{\omega^2}\underline{I} - \underline{\tilde{f}}\,\underline{m}\right]\underline{\hat{v}} = \underline{0}$$

$$\frac{\bar{m}L^4}{9EI}\begin{bmatrix} 42-\lambda & 27 \\ 9 & 6-\lambda \end{bmatrix}\underline{\hat{v}} = \underline{0}$$

λ_1 : $\begin{bmatrix} -5.818 & 27 \\ 9 & -41.818 \end{bmatrix}\begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ for $\hat{v}_{11} = 1$, $\hat{v}_{21} = 0.21522$

λ_2 : $\begin{bmatrix} 41.812 & 27 \\ 9 & 5.8118 \end{bmatrix}\begin{Bmatrix} \hat{v}_{12} \\ \hat{v}_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ for $\hat{v}_{12} = 1$, $\hat{v}_{22} = -1.54857$

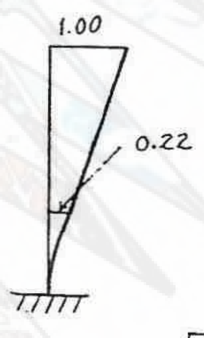
$$\therefore\quad \underline{\hat{v}} = \begin{bmatrix} 1 & 1 \\ 0.21522 & -1.54857 \end{bmatrix}$$

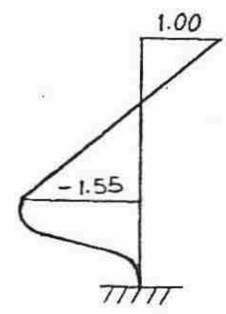
normalizing:

$$\underline{\hat{M}} = \underline{\hat{v}}^T\,\underline{m}\,\underline{\hat{v}} = \bar{m}L\begin{bmatrix} 0.8410 & 0 \\ 0 & 1.2330 \end{bmatrix}$$

$$\underline{\hat{\Phi}} = \underline{\hat{v}}\,\underline{\hat{M}}^{-1/2} = \sqrt{\bar{m}L}\begin{bmatrix} 0.917 & 1.110 \\ 0.1974 & -1.720 \end{bmatrix}$$

$$\frac{1}{\sqrt{\bar{m}L}}$$





$\omega_1 = 0.434\sqrt{\dfrac{EI}{\bar{m}L^4}}$

$\omega_2 = 6.91\sqrt{\dfrac{EI}{\bar{m}L^4}}$

Problem 12-1

Following E12-2, assuming that the effect of damping may be neglected, the modal response-ratio may be expressed by eq. (5-11) (note that the case at hand is a long duration loading).

From eq. (5-11): $$Y_n(t) = \frac{P_{on}}{K_n}(1-\cos\omega_n t) = 2\frac{P_{on}}{K_n}\sin^2\frac{\omega_n t}{2}$$

in which $$K_n = M_n\omega_n^2 = m_n\omega_n^2 \quad \text{(lumped mass)}$$

$$P_{on} = \{\phi_n\}^T\{p_o\}$$

Using the given data:

$$\{K\} = m\{\omega^2\} = 0.4\begin{Bmatrix}(3.61)^2\\(24.2)^2\\(77.2)^2\end{Bmatrix} = \begin{Bmatrix}5.213\\234.3\\2415\end{Bmatrix}\ \text{kips/ft}$$

$$\{P_o\} = [\phi]^T\{p_o\} = [\phi]^T\begin{Bmatrix}0\\p_{20}\\0\end{Bmatrix} = p_{20}\{\phi_{2n}\} = 8\begin{Bmatrix}0.406\\0.870\\-0.281\end{Bmatrix} = \begin{Bmatrix}3.248\\6.96\\-2.248\end{Bmatrix}\ \text{kips}$$

$$\therefore \{Y(t)\} = \begin{Bmatrix}1.246\sin^2(1.805t)\\0.0594\sin^2(12.1t)\\-0.001862\sin^2(38.9t)\end{Bmatrix}\ \text{ft}$$

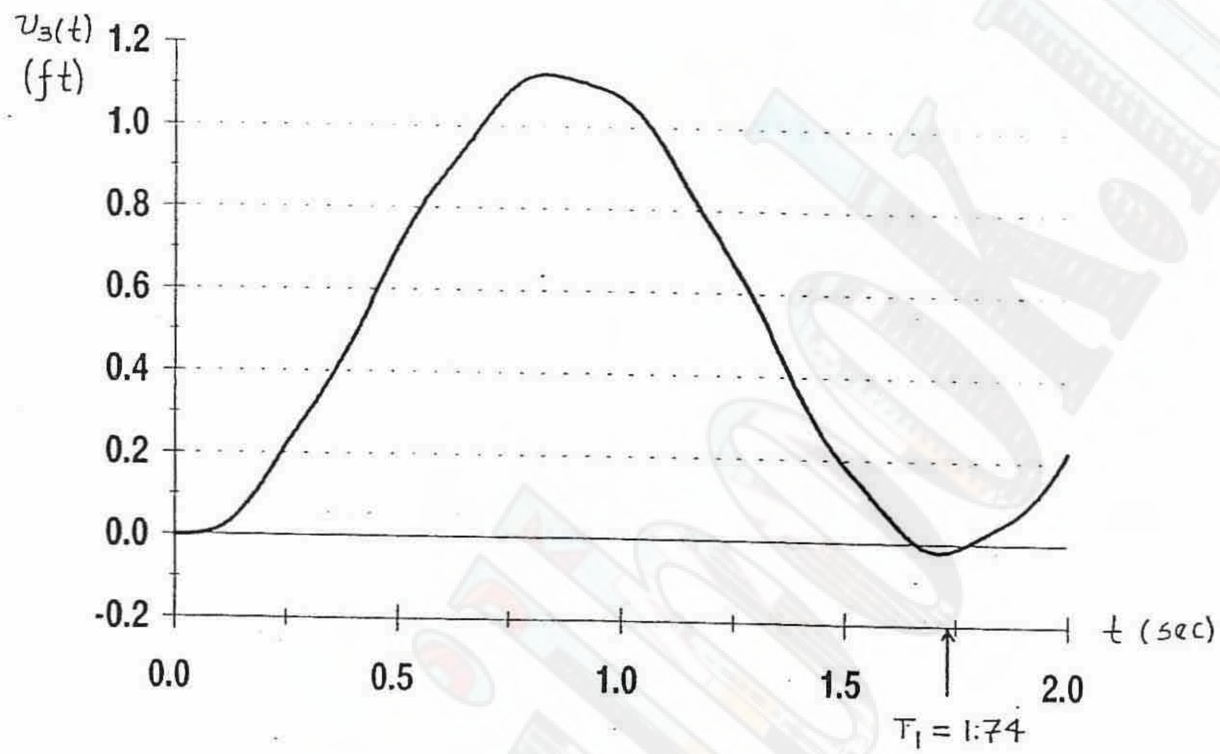
For v_3, $$v_3(t) = \sum_{n=1}^{3}\phi_{3n}Y_n(t) = \langle 0.913, -0.402, 0.068\rangle\begin{Bmatrix}1.246\sin^2(1.805t)\\0.0594\sin^2(12.1t)\\-0.001862\sin^2(38.9t)\end{Bmatrix}$$

$$v_3(t) = 1.138\sin^2(1.805t) - 0.0239\sin^2(12.1t) - 0.0001266\sin^2(38.9t)$$

$$\blacktriangleright\ v_3(t) = 1.14\sin^2(1.81t) - 0.0239\sin^2(12.1t) - 0.000127\sin^2(38.9t)\ [\text{ft}]$$

(continued on following page)

Problem 12-1 (con'd)



Problem 12-2

(a) Following P12-1, from eq. (3-11) and (3-12) :

$$Y_n(t) = \frac{P_{on}}{K_n}\,\frac{1}{1-\beta_n^2}\left(\sin\bar{w}t - \beta_n \sin w_n t\right), \text{ where } \beta_n = \frac{\bar{w}}{w_n}$$

Since

$$\{P_o\} = p_{20}\,\{\phi_{2n}\} = 3\begin{Bmatrix}0.406\\0.870\\-0.281\end{Bmatrix} = \begin{Bmatrix}1.218\\2.61\\-0.843\end{Bmatrix} \text{ kips}$$

Therefore

$$\{Y(t)\} = \begin{Bmatrix} 0.2336\,\dfrac{1}{1-\left(\frac{3}{4}\right)^2}\left(\sin\frac{3}{4}3.61t - \frac{3}{4}\sin 3.61t\right) \\ 0.01114\,\dfrac{1}{1-\left(\frac{3}{4}\cdot\frac{3.61}{24.2}\right)^2}\left(\sin\frac{3}{4}3.61t - \frac{3}{4}\cdot\frac{3.61}{24.2}\sin 24.2t\right) \\ -0.0003491\,\dfrac{1}{1-\left(\frac{3}{4}\cdot\frac{3.61}{77.7}\right)^2}\left(\sin\frac{3}{4}3.61t - \frac{3}{4}\cdot\frac{3.61}{77.7}\sin 77.7t\right) \end{Bmatrix}$$

(continued on following page)

Problem 12-2 (cont'd)

$$\{Y(t)\} = \begin{Bmatrix} 0.5339\,(\sin 2.708t - 0.75 \sin 3.61t) \\ 0.01128\,(\sin 2.708t - 0.1119 \sin 24.2t) \\ -0.0003495\,(\sin 2.708t - 0.03485 \sin 77.7t) \end{Bmatrix} ft$$

For v_1, $v_1(t) = \sum_{n=1}^{3} \phi_{1n} Y_n(t) = \langle 0.054, 0.283, 0.957 \rangle \{Y(t)\}$

$$v_1(t) = 0.03169 \sin 2.708t - 0.02162 \sin 3.61t - 0.0003572 \sin 24.2t + 0.00001166 \sin 77.7t$$

► $v_1(t) = 0.0317 \sin 2.71t - 0.0216 \sin 3.61t - 0.000357 \sin 24.2t + 0.0000117 \sin 77.7t$

(b)

$$\{v(t)\} = [\phi]\{Y(t)\} = \begin{bmatrix} 0.054 & 0.283 & 0.957 \\ 0.406 & 0.870 & -0.281 \\ 0.913 & -0.402 & 0.068 \end{bmatrix} \{Y(t)\}$$

$$\{v(t)\} = \begin{Bmatrix} 0.03169 \sin 2.708t - 0.02162 \sin 3.61t - 0.0003572 \sin 24.2t + 0.00001166 \sin 77.7t \\ 0.2267 \sin 2.708t - 0.1626 \sin 3.61t - 0.001098 \sin 24.2t + 0.000003423 \sin 77.7t \\ 0.4829 \sin 2.708t - 0.3656 \sin 3.61t + 0.0005074 \sin 24.2t + 0.0000008282 \sin 77.7t \end{Bmatrix}$$

(continued on following page)

Problem 12-3

From eq. (3-21), for the steady-state response,

$$Y_n(t) = \rho_n \sin(\bar{\omega} t - \theta_n) \qquad \left(\bar{\omega} = \frac{3}{4}\omega_1\right)$$

where $\rho_n = \dfrac{P_{on}}{K_n}\left[(1-\beta_n^2)^2 + (2\xi_n \beta_n)^2\right]^{-1/2}$, from eq. (3-22)

$$\theta_n = \tan^{-1} \frac{2\xi_n \beta_n}{1-\beta_n^2}, \text{ from eq. (3-23)}$$

$$\beta_n = \frac{\bar{\omega}}{\omega}$$

From P12-1:

$$\{K\} = \begin{Bmatrix} 5.213 \\ 234.3 \\ 2415 \end{Bmatrix} \text{ kips/ft}$$

From P12-2:

$$\{P_o\} = \begin{Bmatrix} 1.218 \\ 2.61 \\ -0.843 \end{Bmatrix} \text{ kips}$$

$$\{\beta\} = \begin{Bmatrix} \frac{3}{4} \\ \frac{3}{4}\cdot\frac{3.61}{24.2} \\ \frac{3}{4}\cdot\frac{3.61}{77.7} \end{Bmatrix} = \begin{Bmatrix} 0.75 \\ 0.1119 \\ 0.03485 \end{Bmatrix}, \quad \{\theta\} = \left\{\tan^{-1} 2(0.1)\frac{\beta_n}{1-\beta_n^2}\right\} = \tan^{-1}\begin{Bmatrix} \frac{0.15}{0.4375} \\ \frac{0.02238}{0.9875} \\ \frac{0.00697}{0.9988} \end{Bmatrix} = \begin{Bmatrix} 18.92^\circ \\ 1.298^\circ \\ 0.3998^\circ \end{Bmatrix}$$

$$\{Y(t)\} = \begin{Bmatrix} 0.5052 \sin(2.708t - 18.92^\circ) \\ 0.01128 \sin(2.708t - 1.298^\circ) \\ -0.00003495 \sin(2.708t - 0.3998^\circ) \end{Bmatrix}$$

For v_1, $v_1(t) = \sum_{n=1}^{3} \phi_{1n} Y_n(t) = \langle 0.054, 0.283, 0.957 \rangle \{Y(t)\}$

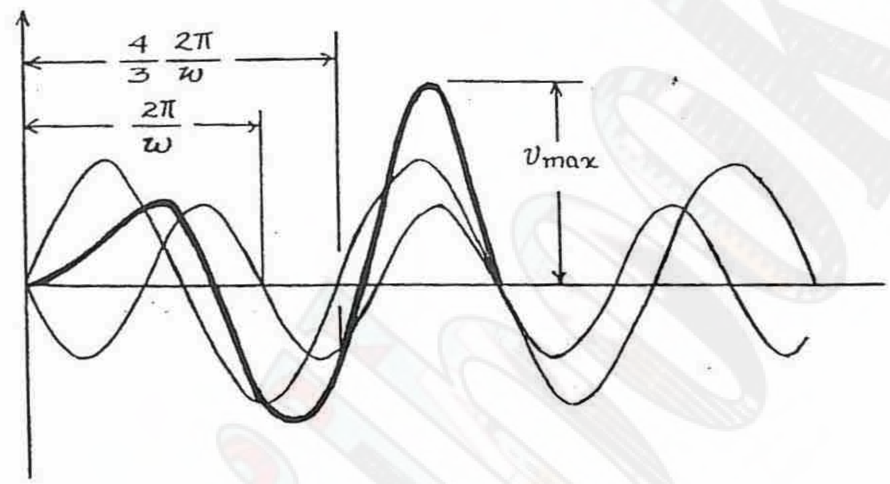
$$v_1(t) = 0.02728 \sin(2.708t - 18.92^\circ) + 0.003192 \sin(2.708t - 1.298^\circ) - 0.00003345 \sin(2.708t - 0.3998^\circ)$$

$$\boxed{v_1(t) = 0.0273 \sin(2.71t - 18.9^\circ) + 0.00319 \sin(2.71t - 1.3^\circ) - 0.0000334 \sin(2.71t - 0.4^\circ)}$$

Problem 12-2 (con'd)

Since the contribution of the higher frecuencies is less than 1.2%, the maximum is almost simultaneous in all masses. The displacement for each mass is goberned by an eq. of the type

$$v(t) = a \sin \frac{3}{4} \omega t - b \sin \omega t$$

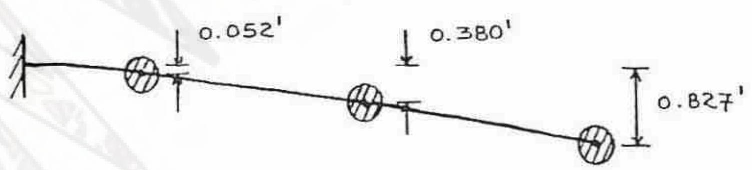


v_{max} is for t in $\frac{4}{3}\frac{2\pi}{\omega} 1.25 < t < \frac{2\pi}{\omega} 1.75$

$$\therefore \quad 1.67 < \frac{\omega t}{2\pi} < 1.75$$

t	1.710	1.711	1.712	1.713	1.714	1.715
$v_1(t)$	0.05197	0.05197	0.05198	0.05198	0.05197	0.05197
$v_2(t)$	0.3795	0.3795	0.3795	0.3795	0.3795	0.3795
$v_3(t)$	0.8270	0.8271	0.8272	0.8272	0.8272	0.8272

$$\blacktriangleright \{v_{max}\} = \{v(t = 1.713)\} = \begin{Bmatrix} 0.0520 \\ 0.380 \\ 0.827 \end{Bmatrix} ft$$



Problem 12-4

eq. (12-3): $\{v\} = [\phi]\{Y\}$

eq. (12-26): $Y_n(t) = e^{-\xi_n \omega_n t}\left[\dfrac{\dot{Y}_n(0) + Y_n(0)\xi_n \omega_n}{\omega_{D_n}} \sin \omega_{D_n} t + Y_n(0) \cos \omega_{D_n} t\right]$

eq. (12-27): $Y_n(0) = \dfrac{1}{M_n}\{\phi_n\}^T [m]\{v(0)\}$ or $\{Y(0)\} = [M]^{-1}[\phi]^T[m]\{v(0)\}$,

eq. (c) of E12-1

eq. (12-28): $\dot{Y}_n(0) = \dfrac{1}{M_n}\{\phi_n\}^T[m]\{\dot{v}(0)\}$

eq. (12-18): $M_n = \{\phi_n\}^T[m]\{\phi_n\}$ or $[M] = [\phi]^T[m][\phi]$, eq. (12-50)

$$\omega_{D_n} = \omega_n\sqrt{1-\xi_n^2}$$

Since $\{v(0)\} = \begin{Bmatrix} 0.3 \\ -0.8 \\ 0.3 \end{Bmatrix}$ in, $\{\dot{v}(0)\} = \{0\}$:

$Y_n(t) = \dfrac{e^{-\xi_n \omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_{D_n} t - \theta_n) Y_n(0)$, $\theta_n = \tan^{-1}\dfrac{\xi_n}{\sqrt{1-\xi_n^2}}$, see eqs. (2-50) to (2-52)

However $\{Y(0)\} = [M]^{-1}[\phi]^T[m]\{v(0)\} = [[\phi]^T[m][\phi]]^{-1}[\phi]^T[m]\{v(0)\}$

$\{Y(0)\} = [\phi]^{-1}[[\phi]^T[m]]^{-1}[[\phi]^T[m]]\{v(0)\} = [\phi]^{-1}\{v(0)\}$

(a) $\xi_n = 0$, $\forall_n \longrightarrow Y_n(t) = Y(0)\cos\omega t$

$$\{Y(t)\} = \begin{Bmatrix} -0.05623 \cos 11.62t \\ 0.30830 \cos 27.5t \\ 0.04789 \cos 45.9t \end{Bmatrix} \text{ in}$$

$$\left\{Y\left(\frac{2\pi}{\omega_1} = 0.5407\right)\right\} = \begin{Bmatrix} -0.05623 \\ -0.20607 \\ 0.04554 \end{Bmatrix} \text{ in}$$

(continued on following page)

Problem 12-4 (con'd)

$$\therefore \left\{v\left(t=\frac{2\pi}{\omega_1}\right)\right\} = [\phi]\left\{Y\left(t=\frac{2\pi}{\omega_1}\right)\right\} = \begin{Bmatrix} -0.21676 \\ -0.00428 \\ 0.71959 \end{Bmatrix} \text{ in}$$

$$\blacktriangleright \left\{v\left(\frac{2\pi}{t_1}\right)\right\} = \begin{Bmatrix} -0.21800 \\ -0.00428 \\ 0.72000 \end{Bmatrix} \text{ in}$$

(b) $\xi_n = \xi = 0.1$, $\forall_n \rightarrow \omega_{D_n} = 0.9950\,\omega_n$

$$\{Y(t)\} = \begin{Bmatrix} -0.05651 \; e^{-0.1162t} \cos(11.56t - \theta) \\ 0.30980 \; e^{-2.75t} \cos(27.36t - \theta) \\ 0.04813 \; e^{-4.59t} \cos(45.67t - \theta) \end{Bmatrix} \text{ in}, \; \theta = \tan^{-1} 0.1005 = 5.74^\circ$$

$$\left\{Y\left(\frac{2\pi}{\omega_1} = 0.5407\right)\right\} = \begin{Bmatrix} -0.05260 \\ -0.03698 \\ 0.003453 \end{Bmatrix} \text{ in}$$

$$\therefore \left\{v\left(t=\frac{2\pi}{\omega_1}\right)\right\} = [\phi]\left\{Y\left(t=\frac{2\pi}{\omega_1}\right)\right\} = \begin{Bmatrix} -0.08613 \\ 0.00584 \\ 0.06361 \end{Bmatrix} \text{ in}$$

$$\blacktriangleright \left\{v\left(\frac{2\pi}{\omega_1}\right)\right\} = \begin{Bmatrix} -0.08610 \\ 0.00584 \\ 0.06360 \end{Bmatrix} \text{ in}$$

Problem 12-5

Eq. (12-3): $\{v\} = [\phi]\{Y\}$

From eq. (3-21): $Y_n(t) = \rho_n \sin(\bar{\omega} t - \theta_n)$

eq. (3-22): $\rho_n = \frac{P_{on}}{k_n}\left[\left(1-\beta_n^2\right)^2 + \left(2\xi_n\beta_n\right)^2\right]^{-1/2}$

eq. (3-23): $\theta_n = \tan^{-1}\frac{2\xi_n\beta_n}{1-\beta_n^2}$, $\beta_n = \frac{\bar{\omega}}{\omega_n}$

eq. (12-12c): $P_{on} \equiv \{\phi_n\}^T\{p_o\}$ or from eq. (c), E12-2: $\{P_o\} = [\phi]^T\{p_o\}$

eq. (12-12d): $K_n = \omega_n^2 M_n$

eq. (12-12a): $M_n \equiv \{\phi_n\}^T[m]\{\phi_n\}$ or eq. (12-18): $[M] = [\phi]^T[m][\phi]$

$$\therefore \{P_o\} = [\phi]^T\begin{Bmatrix} p_{01} \\ 0 \\ 0 \end{Bmatrix} = p_{01}\{\phi_1\} = 5\begin{Bmatrix} 1 \\ 0.548 \\ 0.198 \end{Bmatrix} = \begin{Bmatrix} 5.00 \\ 2.74 \\ 0.99 \end{Bmatrix} \text{ kips}$$

$$[M] = [\phi]^T mI[\phi] = m[\phi]^T[\phi] = 2\left[\diagdown \sum_{i=1}^{3}\phi_{in}^2 \diagdown\right] = 2\begin{bmatrix} 1.340 & & \\ & 4.077 & \\ & & 186.6 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 2.680 & & \\ & 8.154 & \\ & & 373.2 \end{bmatrix} \text{ kips}\cdot\text{sec}^2/\text{in}$$

$$\{K\} = \{\omega^2 M\} = \begin{Bmatrix} (11.62)^2 \; 2.680 \\ (27.5)^2 \; 8.154 \\ (45.9)^2 \; 373.2 \end{Bmatrix} = \begin{Bmatrix} 361.9 \\ 6166 \\ 786300 \end{Bmatrix} \text{ kips/in}$$

$$\{\beta\} = \begin{Bmatrix} 1.1 \\ 1.1\,\frac{11.62}{27.5} \\ 1.1\,\frac{11.62}{45.9} \end{Bmatrix} = \begin{Bmatrix} 1.1 \\ 0.4648 \\ 0.2785 \end{Bmatrix}$$

(continued on following page)

Problem 12-5 (con'd)

$$\therefore \{\rho\} = \begin{Bmatrix} 0.01382\left[(0.1)^2 + (0.22)^2\right]^{-1/2} \\ 0.0004444\left[(0.5352)^2+(0.09296)^2\right]^{-1/2} \\ 0.000001259\left[(0.7215)^2+(0.0557)^2\right]^{-1/2} \end{Bmatrix} = \begin{Bmatrix} 5.719\times10^{-2} \\ 8.181\times10^{-4} \\ 1.740\times10^{-6} \end{Bmatrix} \text{ in}$$

$$\{\theta\} = \tan^{-1}\begin{Bmatrix} -\dfrac{0.22}{0.1} \\ \dfrac{0.09296}{0.5352} \\ \dfrac{0.0557}{0.7215} \end{Bmatrix} = \begin{Bmatrix} -65.56^\circ \\ 9.854^\circ \\ 4.414^\circ \end{Bmatrix}$$

Neglecting the contribution of the two higher modes : $\{v\} = \{\phi_1\} Y_1$

Then, the amplitude will be: $\{\rho_v\} = \begin{Bmatrix} 1 \\ 0.584 \\ 0.198 \end{Bmatrix} 5.719\times10^{-2} = \begin{Bmatrix} 5.719 \\ 3.340 \\ 1.132 \end{Bmatrix}\times10^{-2} \text{ in}$

and the phase angle will be common, $\theta = -65.56^\circ$

However, if the contribution of all modes is considered:

$$\{v(t)\} = \begin{bmatrix} 1 & 1 & 1 \\ 0.584 & -1.522 & -6.26 \\ 0.198 & -0.872 & 12.1 \end{bmatrix} \begin{Bmatrix} 5.719\times10^{-2}\sin(\bar{\omega}t + 65.56^\circ) \\ 8.181\times10^{-4}\sin(\bar{\omega}t - 9.854^\circ) \\ 1.740\times10^{-6}\sin(\bar{\omega}t - 4.414^\circ) \end{Bmatrix}$$

$$\{v(t)\} = \begin{bmatrix} 1 & 1 & 1 \\ 0.584 & -1.522 & -6.26 \\ 0.198 & -0.872 & 12.1 \end{bmatrix} \begin{Bmatrix} 2.366\times10^{-2}\sin\bar{\omega}t + 5.207\times10^{-2}\cos\bar{\omega}t \\ 8.060\times10^{-4}\sin\bar{\omega}t - 1.400\times10^{-4}\cos\bar{\omega}t \\ 1.735\times10^{-6}\sin\bar{\omega}t - 0.1339\times10^{-6}\cos\bar{\omega}t \end{Bmatrix}$$

$$\{v(t)\} = \begin{Bmatrix} 2.448\sin\bar{\omega}t + 5.193\cos\bar{\omega}t \\ 1.258\sin\bar{\omega}t + 3.062\cos\bar{\omega}t \\ 0.4005\sin\bar{\omega}t + 1.043\cos\bar{\omega}t \end{Bmatrix}\times10^{-2}\text{ in} \rightarrow \{\rho_v\} = \begin{Bmatrix} 5.741 \\ 3.310 \\ 1.117 \end{Bmatrix}\times10^{-2}\text{ in},\ \theta = \begin{Bmatrix} -64.76^\circ \\ -67.67^\circ \\ -69.00^\circ \end{Bmatrix}$$

$$\blacktriangleright \{\rho\} = \begin{Bmatrix} 0.0574 \\ 0.0331 \\ 0.0112 \end{Bmatrix}\text{ in},\ \theta = \begin{Bmatrix} -64.8^\circ \\ -67.7^\circ \\ -69.0^\circ \end{Bmatrix}$$

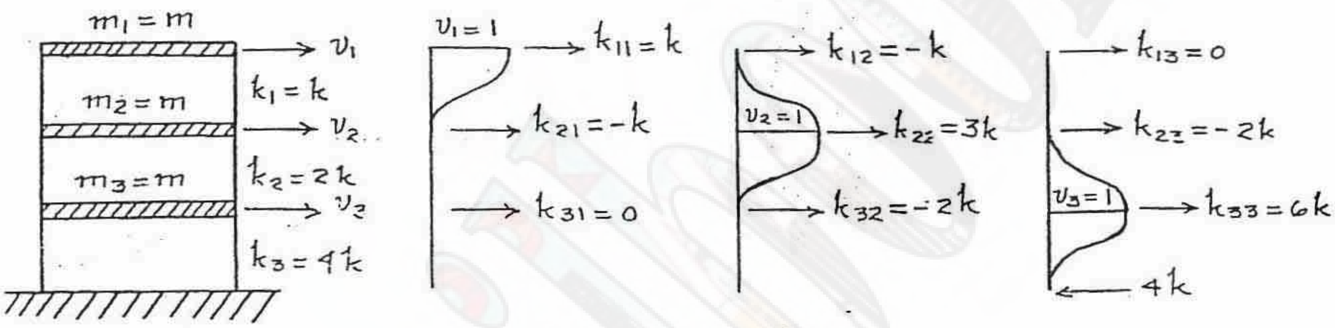
Problem 12-6

Following E12-4:

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} = \begin{Bmatrix} 0.05 \\ 0.15 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{11.62} & 11.62 \\ \frac{1}{45.9} & 45.9 \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \longrightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 0.2986 \ \text{1/sec} \\ 0.006394 \ \text{sec} \end{Bmatrix}$$

$$\therefore [c] = 0.2986[m] + 0.006394[k]$$

Direct formulation of the mass and stiffness matrices:



$$[m] = mI, \quad m = 2 \ \text{kips}\cdot\text{sec}^2/\text{in} \qquad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix}, \quad k = 600 \ \text{kips/in}$$

$$\therefore [c] = 0.2986(2I) + 0.006394(600)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 4.434 & -3.836 & 0 \\ -3.836 & 12.106 & -7.673 \\ 0 & -7.673 & 23.616 \end{bmatrix} \frac{\text{kips}\cdot\text{sec}}{\text{in}}$$

Also, $\xi_2 = \frac{1}{2} \left\langle \frac{1}{27.5}, \ 27.5 \right\rangle \begin{Bmatrix} 0.2986 \\ 0.006394 \end{Bmatrix} = 0.09334$

$$\blacktriangleright [c] = \begin{bmatrix} 4.43 & -3.84 & 0 \\ -3.84 & 12.11 & -7.67 \\ 0 & -7.67 & 23.62 \end{bmatrix} \frac{\text{kips}\cdot\text{sec}}{\text{in}} \ ; \ \xi_2 = 9.33\%$$

Problem 12-7

Eq. (12-57a) : $a_1 = \dfrac{2\xi_c}{\omega_c}$

eq. (12-57c) : $\bar{\xi}_n = \xi_n - \xi_c\left(\dfrac{\omega_n}{\omega_c}\right)$

eq. (12-57d) : $[c] = a_1[k] + [m]\left[\sum_{n=1}^{c-1} \dfrac{2\bar{\xi}_n \omega_n}{M_n}\{\phi_n\}\{\phi_n\}^T\right][m]$

From P12-6 :

$$[m] = mI\ ,\ m = 2\ \text{kips}\cdot\text{sec}^2/\text{in} \text{ and } [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix},\ k = 600\frac{\text{kips}}{\text{in}}$$

From P12-5 :

$$\{M\} = \begin{Bmatrix} 2.680 \\ 8.154 \\ 373.2 \end{Bmatrix} \text{kips}\cdot\text{sec}^2/\text{in}$$

$$\therefore [c] = \frac{2\xi_3}{\omega_3}[k] + m^2\sum_{n=1}^{2}\frac{2\bar{\xi}_n\omega_n}{M_n}\{\phi_n\}\{\phi_n\}^T$$

$$[c] = \frac{2(0.12)}{45.9}\,600\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix} + 2^2\left[\frac{2\left(0.08 - 0.12\frac{11.62}{45.9}\right)11.62}{2.680}\begin{Bmatrix} 1 \\ 0.548 \\ 0.198 \end{Bmatrix}\cdot\right.$$

$$\left. \langle 1, 0.548, 0.198 \rangle + \frac{2\left(0.10 - 0.12\frac{27.5}{45.9}\right)27.5}{8.154}\begin{Bmatrix} 1 \\ -1.522 \\ -0.872 \end{Bmatrix}\langle 1, -1.522, -0.872 \rangle\right]$$

$$[c] = 3.1373\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix} + 1.7212\begin{bmatrix} 1 & 0.548 & 0.198 \\ 0.548 & 0.300 & 0.109 \\ 0.198 & 0.109 & 0.0392 \end{bmatrix}$$

$$+ 0.1896\begin{bmatrix} 1 & -1.522 & -0.872 \\ -1.522 & 2.316 & 1.327 \\ -0.872 & 1.327 & 0.760 \end{bmatrix}$$

(continued on following page)

Problem 12-7 (con'd)

$$[c] = \begin{bmatrix} 5.048 & -2.483 & 0.175 \\ -2.483 & 10.367 & -5.835 \\ 0.175 & -5.835 & 19.035 \end{bmatrix} \text{ kips} \cdot \text{sec/in}$$

$$\blacktriangleright [c] = \begin{bmatrix} 5.050 & -2.480 & 0.175 \\ -2.480 & 10.400 & -5.840 \\ 0.175 & -5.840 & 19.000 \end{bmatrix} \text{ kips} \cdot \text{sec/in}$$

Problem 13-1

From P 11-1:

$$[m] = m I,\ m = 2\ \text{kips}\cdot\text{sec}^2/\text{in}; \quad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix},\ k = 400\ \text{kips}/\text{in}$$

Following E13-1:

$$[\tilde{f}] = [k]^{-1} = \frac{1}{6k}\begin{bmatrix} 11 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$[D] = [\tilde{f}][m] = m[\tilde{f}] = \frac{m}{6k}\begin{bmatrix} 11 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{cccccccccc} [D] & \{v_1^{(0)}\} & & \{\bar{v}_1^{(1)}\} & \{v_1^{(1)}\} & \{\bar{v}_1^{(2)}\} & \{v_1^{(2)}\} & \{\bar{v}_1^{(3)}\} & \{v_1^{(3)}\} & \{\bar{v}_1^{(4)}\} \\ \frac{m}{6k}\begin{bmatrix} 11 & 5 & 2 \\ 5 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} & \begin{Bmatrix} 1 \\ 0.6667 \\ 0.3333 \end{Bmatrix} & = & \begin{Bmatrix} 15 \\ 9 \\ 4 \end{Bmatrix} & \begin{matrix} 1 \\ 0.6000 \\ 0.2667 \end{matrix} & \begin{matrix} 14.5334 \\ 8.5334 \\ 3.7334 \end{matrix} & \begin{matrix} 1 \\ 0.5872 \\ 0.2569 \end{matrix} & \begin{matrix} 14.4498 \\ 8.4498 \\ 3.6882 \end{matrix} & \begin{matrix} 1 \\ 0.5848 \\ 0.2552 \end{matrix} & \begin{matrix} 14.4344 \\ 8.4344 \\ 3.6800 \end{matrix} \end{array}$$

$$\begin{array}{cc} \{v_1^{(4)}\} & \{\bar{v}_1^{(5)}\} \\ \begin{matrix} 1 \\ 0.5843 \\ 0.2549 \end{matrix} & \begin{matrix} 14.4313 \\ \\ \\ \end{matrix} \end{array}$$

final shape

$$\therefore\ \omega_1^2 = \frac{v_{11}^{(4)}}{\bar{v}_{11}^{(5)}} = \frac{1}{\frac{m}{6k}14.4313} = 0.4158\frac{k}{m} \longrightarrow \omega_1 = 0.6448\sqrt{\frac{k}{m}} = 9.119\ 1/\text{sec}$$

$$\blacktriangleright \{\hat{v}_1\} = \begin{Bmatrix} 1 \\ 0.584 \\ 0.255 \end{Bmatrix},\quad \omega_1 = 9.12\ 1/\text{sec}$$

Problem 13-2

From P11-1:

$$[m] = m\,I,\quad m = 2\ \text{kips}\cdot\text{sec}^2/\text{in},\quad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix},\quad k = 400\ \text{kips/in}$$

Following E13-3: $[m]^{-1} = \dfrac{1}{m} I$

Eq.(13-45):

$$[D]^{-1} = [E] = [m]^{-1}[k] = \frac{k}{m}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\underbrace{\frac{k}{m}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}}_{[E]}\underbrace{\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}}_{\{v_3^{(0)}\}} = \underbrace{\begin{Bmatrix} 2 \\ -6 \\ 7 \end{Bmatrix}}_{\{\bar{v}_3^{(1)}\}}$$

$\{v_3^{(1)}\}$	$\{\bar{v}_3^{(2)}\}$	$\{v_3^{(2)}\}$	$\{\bar{v}_3^{(3)}\}$	$\{v_3^{(3)}\}$	$\{\bar{v}_3^{(4)}\}$	$\{v_3^{(4)}\}$	$\{\bar{v}_3^{(5)}\}$
0.2857	1.1428	0.1702	0.8936	0.1386	0.8119	0.1279	0.7831
−0.8571	−4.8570	−0.7234	−4.3404	−0.6733	−4.1585	−0.6552	−4.0935
1	6.7142	1	6.4468	1	6.3466	1	6.3104

$\{v_3^{(5)}\}$	$\{\bar{v}_3^{(6)}\}$	$\{v_3^{(6)}\}$	$\{\bar{v}_3^{(7)}\}$	$\{v_3^{(7)}\}$	$\{\bar{v}_3^{(8)}\}$
0.1241	0.7728	0.1227	0.7690	0.1222	
−0.6487	−4.0702	−0.6463	−4.0616	−0.6454	
1	6.2974	1	6.2926	1	6.2908

final shape ($\{v_3^{(7)}\}$)

$$\therefore\ \omega_3^2 = \frac{\bar{v}_{33}^{(8)}}{v_{33}^{(7)}} = \frac{\frac{k}{m}\,6.2908}{1} = 6.2908\,\frac{k}{m}$$

$$\therefore\ \omega_3 = 2.508\sqrt{\frac{k}{m}} = 35.17\ 1/\text{sec}$$

$$\blacktriangleright\ \{\hat{v}_3\} = \begin{Bmatrix} 0.122 \\ -0.645 \\ 1 \end{Bmatrix},\quad \omega_3 = 35.5\ 1/\text{sec}$$

Problem 13-3

From P11-2:

$$[m] = m\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}, \quad m = 1 \ \text{kips}\cdot\text{sec}^2/\text{in}; \quad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, k = 800\ \frac{\text{kips}}{\text{in}}$$

Following P13-1:

$$[\tilde{f}] = \frac{1}{k}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[D] = \frac{m}{k}\begin{bmatrix} 3 & 4 & 3 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$[D]$	$\{v_1^{(0)}\}$	$\{\bar{v}_1^{(1)}\}$	$\{v_1^{(1)}\}$	$\{\bar{v}_1^{(2)}\}$	$\{v_1^{(2)}\}$	$\{\bar{v}_1^{(3)}\}$	$\{v_1^{(3)}\}$	$\{\bar{v}_1^{(4)}\}$
$\frac{m}{k}$ [3 4 3]	1	6.6668	1	7.45	1	7.9734	1	8.0944
[2 4 3]	0.6667	5.6668	0.85	6.45	0.8658	6.9734	0.8746	7.0944
[1 2 3]	0.3333	2.3334	0.35	3.75	0.5034	4.2418	0.5320	4.3452

$\{v_1^{(4)}\}$	$\{\bar{v}_1^{(5)}\}$	$\{v_1^{(5)}\}$	$\{\bar{v}_1^{(6)}\}$
1	8.1164	1	8.12
0.8765	7.1164	0.8768	
0.5368	4.3634	0.5376	

final shape

$$\omega_1^2 = \frac{1}{\frac{m}{k}8.12} = 0.1232\,\frac{k}{m} \longrightarrow \omega_1 = 0.3509\sqrt{\frac{k}{m}} = 9.926\ 1/\text{sec}$$

$$\{\hat{v}_1\} = \begin{Bmatrix} 1 \\ 0.877 \\ 0.538 \end{Bmatrix}, \quad \omega_1 = 9.93\ 1/\text{sec}$$

Problem 13-4

From P12-6 :

$$[m] = m\mathrm{I}, \quad m = 2 \text{ kips}\cdot\text{sec}^2/\text{in}; \qquad [k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix}, \quad k = 600 \text{ kips/in}$$

From P 12-5 : $M_1 = 1.340\ m$

From P12-7 :

$$\{\phi_1\}\{\phi_1\}^T = \begin{bmatrix} 1 & 0.548 & 0.198 \\ 0.548 & 0.300 & 0.109 \\ 0.198 & 0.109 & 0.0392 \end{bmatrix}$$

Following E13-2 :

$$[S_1] = \mathrm{I} - \frac{1}{M_1}\{\phi_1\}\{\phi_1\}^T[m] = \mathrm{I} - \frac{1}{1.340m}\begin{bmatrix} 1 & 0.548 & 0.198 \\ 0.548 & 0.300 & 0.109 \\ 0.198 & 0.109 & 0.0392 \end{bmatrix} m\mathrm{I}$$

$$[S_1] = \begin{bmatrix} 0.25373 & -0.40896 & -0.14776 \\ -0.40896 & 0.77612 & -0.081343 \\ -0.14776 & -0.081343 & 0.97075 \end{bmatrix}$$

$$[D_2] = [D][S_1] = [\tilde{f}][m][S_1] = [k]^{-1}[m][S_1] = k^{-1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix}^{-1} m\mathrm{I}[S_1]$$

$$[D_2] = \frac{m}{k}\,\frac{1}{4}\begin{bmatrix} 7 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.25373 & -0.40896 & -0.14776 \\ -0.40896 & 0.77612 & -0.081343 \\ -0.14776 & -0.081343 & 0.97075 \end{bmatrix}$$

$$[D_2] = \frac{m}{4k}\begin{bmatrix} 0.40147 & -0.61570 & -0.30760 \\ -0.61345 & 1.02014 & 0.28344 \\ -0.30299 & 0.28582 & 0.74165 \end{bmatrix}$$

$$\begin{array}{ccccc} [D_2] & \{v_2^{(0)}\} & \{\bar{v}_2^{(1)}\} & \{v_2^{(1)}\} & \{\bar{v}_2^{(2)}\} \\ \frac{m}{4k}\begin{bmatrix} 0.40147 & -0.61570 & -0.30760 \\ -0.61345 & 1.02014 & 0.28344 \\ -0.30299 & 0.28582 & 0.74165 \end{bmatrix} & \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix} & \begin{matrix} 1.32477 \\ -1.91703 \\ -1.33046 \end{matrix} & \begin{Vmatrix} 1 \\ -1.44707 \\ -1.00430 \end{Vmatrix} & \begin{matrix} 1.60135 \\ -2.3743 \\ -1.46143 \end{matrix} \end{array}$$

(continued on following page)

Problem 13-4 (con'd)

$\{v_2^{(2)}\}$	$\{\bar{v}_2^{(3)}\}$	$\{v_2^{(3)}\}$	$\{\bar{v}_2^{(4)}\}$	$\{v_2^{(4)}\}$	$\{\bar{v}_2^{(5)}\}$	$\{v_2^{(5)}\}$	$\{\bar{v}_2^{(6)}\}$
1	1.59509	1	1.59262	1	1.59175	1	1.59143
−1.48270	−2.3847	−1.49501	−2.3880	−1.49940	−2.3892	1.50097	
−0.91262	−1.40362	−0.87996	−1.38292	−0.86833	−1.37555	−0.86417	

final shape

$$\omega_2^2 = \frac{v_{21}^{(5)}}{\bar{v}_{21}^{(6)}} = \frac{1}{\frac{m}{4k}\,1.59143} = 2.5135\,\frac{k}{m}$$

$$\omega_2 = 1.5854\sqrt{\frac{k}{m}} = 27.460\ {}^{1}/_{sec}$$

$$\blacktriangleright \{\hat{v}_2\} = \begin{Bmatrix} -0.666 \\ 1 \\ 0.576 \end{Bmatrix}, \quad \omega_2 = 27.5\ {}^{1}/_{sec}$$

Problem 13-5

Eq. (13-65): $[\hat{E}][\Phi] = [\Phi]\{\hat{\delta}\}$

eq. (13-64): $[[E] - \mu I][\Phi] = [\Phi]\{\hat{\delta}\}$ $\longrightarrow [\hat{E}] = [m]^{-1}[k] - \mu[I]$

eq. (13-45): $[E] = [m]^{-1}[k]$

Eq. (13-65) for the mode n: $[\hat{E}]\{\phi_n\} = \delta_n\{\phi_n\}$

or, assuming $\{v_n^{(0)}\}$: $[\hat{E}]\{v_n^{(0)}\} = \{\bar{v}_n^{(1)}\} = \omega_n^2\{v_n^{(1)}\}$ analogous to eq. (13-46)

by inverse iteration: $[\hat{E}]\{\bar{v}_n^{(1)}\} = \{v_n^{(0)}\}$

$$\therefore \{\bar{v}_n^{(1)}\} = [\hat{E}]^{-1}\{v_n^{(0)}\}$$

(continued on following page)

Problem 13-5 (con'd)

From P12-6 : $[m] = mI$, $m = 2 \ \text{kips}\cdot\text{sec}^2/\text{in}$; $[k] = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix}$, $k = 600 \frac{\text{kips}}{\text{in}}$

$$\therefore [\hat{E}] = m^{-1} I \, k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix} - \mu I = \frac{k}{m}\left[\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix} - \mu \frac{m}{k} I\right]$$

since $\mu \frac{m}{k} = 0.98 \, \omega_2^2 \, \frac{m}{k} = 0.98 (27.5)^2 \frac{2}{600} = 741.1 \frac{1}{300} = 2.470$,

$$\therefore [\hat{E}] = \frac{k}{m}\begin{bmatrix} -1.470 & -1 & 0 \\ -1 & 0.530 & -2 \\ 0 & -2 & 3.530 \end{bmatrix} \longrightarrow [\hat{E}]^{-1} = \frac{m}{k}\begin{bmatrix} 5.319 & -8.818 & -4.996 \\ -8.818 & 12.962 & 7.344 \\ -4.996 & 7.344 & 4.444 \end{bmatrix}$$

$$\begin{array}{ccccccc} [\hat{E}]^{-1} & \{v_2^{(0)}\} & \{\bar{v}_2^{(1)}\} & \{v_2^{(1)}\} & \{\bar{v}_2^{(2)}\} & \{v_2^{(2)}\} & \{\bar{v}_2^{(3)}\} \\ \frac{m}{k}\begin{bmatrix} 5.319 & -8.818 & -4.996 \\ -8.818 & 12.962 & 7.344 \\ -4.996 & 7.344 & 4.444 \end{bmatrix} & \begin{matrix} -0.5 \\ 1 \\ 0.5 \end{matrix} & \begin{matrix} -14.0 \\ 21.1 \\ 12.1 \end{matrix} & \begin{matrix} -0.664 \\ 1 \\ 0.573 \end{matrix} & \begin{matrix} -15.21 \\ 23.03 \\ 13.21 \end{matrix} & \begin{matrix} -0.6604 \\ 1 \\ 0.5736 \end{matrix} & \begin{matrix} -15.196 \\ 22.998 \\ 13.192 \end{matrix} \end{array}$$

$$\begin{array}{ccc} \{v_2^{(3)}\} & \{\bar{v}_2^{(4)}\} & \\ \begin{matrix} -0.6608 \\ 1 \\ 0.5736 \end{matrix} & \begin{matrix} -15.199 \\ 23.001 \\ 13.194 \end{matrix} & \begin{matrix} -0.6608 \\ 1 \\ 0.5736 \end{matrix} \end{array}$$

final shape

From eq. (13-68): $\omega_2^2 \equiv \mu + \frac{1}{\bar{v}_{22}^{(4)}} = 741.1 + \frac{k}{23.001\,m} = 754.1 \ ^1/\text{sec}^2$

$$\therefore \omega_2 = 27.46 \ ^1/\text{sec}$$

$$\blacktriangleright \{\hat{v}_2\} = \begin{Bmatrix} -0.661 \\ 1 \\ 0.574 \end{Bmatrix}, \quad \omega_2 = 27.5 \ ^1/\text{sec}$$

Problem 13-6

Following E13-4 :

$$[k_G] = \begin{bmatrix} \frac{N_0}{l_0} + \frac{N_1}{l_1} & -\frac{N_1}{l_1} & 0 \\ -\frac{N_1}{l_1} & \frac{N_1}{l_1} + \frac{N_2}{l_2} & -\frac{N_2}{l_2} \\ 0 & -\frac{N_2}{l_2} & \frac{N_2}{l_2} \end{bmatrix} = \begin{bmatrix} \frac{N}{L/3} + \frac{N}{L/3} & -\frac{N}{L/3} & 0 \\ -\frac{N}{L/3} & \frac{N}{L/3} + \frac{N}{2L/3} & -\frac{N}{2L/3} \\ 0 & -\frac{N}{2L/3} & \frac{N}{2L/3} \end{bmatrix}$$

$$[k_G] = \frac{3N}{2L}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[G] = [\tilde{f}][k_{G_0}] = [\tilde{f}][k_G] = \frac{L^3}{243EI}\begin{bmatrix} 8 & 7 & -8 \\ 7 & 8 & -10 \\ -8 & -10 & 24 \end{bmatrix}\frac{3N}{2L}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[G] = \frac{NL^2}{162EI}\begin{bmatrix} 18 & 13 & -15 \\ 12 & 20 & -18 \\ -12 & -38 & 34 \end{bmatrix}$$

$$\begin{array}{cccccc} [G] & \{v_1^{(0)}\} & \{\bar{v}_1^{(1)}\} & \{v_1^{(1)}\} & \{\bar{v}_1^{(2)}\} & \{v_1^{(2)}\} \quad \{\bar{v}_1^{(3)}\} \end{array}$$

$$\frac{NL^2}{162EI}\begin{bmatrix} 18 & 13 & -15 \\ 12 & 20 & -18 \\ -12 & -38 & 34 \end{bmatrix}\left\|\begin{matrix} -0.5 & -31 \\ -0.5 & -34 \\ 1 & 59 \end{matrix}\right\|\begin{matrix} -0.525 & -31.94 \\ -0.576 & -35.82 \\ 1 & 62.19 \end{matrix}\left\|\begin{matrix} -0.5136 & -31.73 \\ -0.5760 & -35.68 \\ 1 & 62.05 \end{matrix}\right|$$

$$\begin{array}{cccc} \{v_1^{(3)}\} & \{\bar{v}_1^{(4)}\} & \{v_1^{(4)}\} & \{\bar{v}_1^{(5)}\} \end{array}$$

$$\left|\begin{matrix} -0.5114 & -31.68 \\ -0.5750 & -35.64 \\ 1 & 61.99 \end{matrix}\right\|\begin{matrix} -0.5110 & \\ -0.5749 & \\ 1 & 61.98 \end{matrix}\right|$$

final shape

$$\lambda_{cr} = \frac{v_{31}^{(4)}}{\bar{v}_{31}^{(5)}} = \frac{1}{61.98\,\frac{NL^2}{162EI}} = 2.614\,\frac{EI}{NL^4}, \qquad \therefore\ N_{cr} = \lambda_{cr} N = 2.614\,\frac{EI}{L^2}$$

$$\blacktriangleright\ \boxed{N_{cr} = 2.61\,\frac{EI}{L^2}}$$

Problem 13-7

From P13-6:

$$[k_G] = \frac{3N}{2L}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[\bar{k}] = [k] - [k_G] = \frac{243EI}{168L^3}\begin{bmatrix} 92 & -88 & -6 \\ -88 & 128 & 24 \\ -6 & 24 & 15 \end{bmatrix} - \frac{3EI}{L^3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[\bar{k}] = \frac{3EI}{56L^3}\begin{bmatrix} 2260 & -2264 & -162 \\ -2264 & 3288 & 704 \\ -162 & 704 & 349 \end{bmatrix}$$

Lumped mass matrix:

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Following E13-1:

$$[\tilde{f}] = [k]^{-1} = \frac{1}{766983}\begin{bmatrix} 81487 & 84511 & -132650 \\ 84511 & 95312 & -153034 \\ -132650 & -153034 & 288148 \end{bmatrix}\frac{L^3}{EI}$$

$$[\tilde{f}] = [k]^{-1} = \begin{bmatrix} 0.10624 & 0.11019 & -0.17295 \\ 0.11019 & 0.12427 & -0.19953 \\ -0.17295 & -0.19953 & 0.37569 \end{bmatrix}\frac{L^3}{EI}$$

$$[D] = [\tilde{f}][m] = \frac{mL^3}{EI}\begin{bmatrix} 0.10624 & 0.11019 & -0.34590 \\ 0.11019 & 0.12427 & -0.39906 \\ -0.17295 & -0.19953 & 0.75138 \end{bmatrix}$$

$$\begin{array}{cccccccc} & [D] & \{v_1^{(0)}\} & \{\bar{v}_1^{(1)}\} & \{v_1^{(1)}\} & \{\bar{v}_1^{(2)}\} & \{v_1^{(2)}\} & \{\bar{v}_1^{(3)}\} \\ \frac{mL^3}{EI} & \begin{bmatrix} 0.10624 & 0.11019 & -0.34590 \\ 0.11019 & 0.12427 & -0.39906 \\ -0.17295 & -0.19953 & 0.75138 \end{bmatrix} & \begin{matrix} -0.5 \\ -0.5 \\ 1 \end{matrix} & \begin{matrix} -0.454 \\ -0.516 \\ 0.938 \end{matrix} & \begin{matrix} -0.484 \\ -0.550 \\ 1 \end{matrix} & \begin{matrix} -0.4579 \\ -0.5207 \\ 0.9448 \end{matrix} & \begin{matrix} -0.4847 \\ -0.5511 \\ 1 \end{matrix} & \begin{matrix} -0.4581 \\ -0.5210 \\ 0.9452 \end{matrix} \end{array}$$

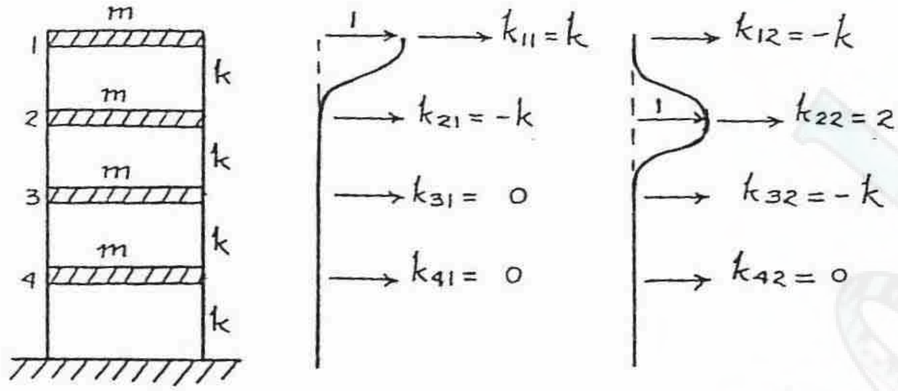
$$\begin{array}{cc} \{v_1^{(3)}\} & \{\bar{v}_1^{(4)}\} \\ \begin{matrix} -0.4846 \\ -0.5512 \\ 1 \end{matrix} & \begin{matrix} \\ \\ 0.9452 \end{matrix} \end{array}$$

$$\omega_1^2 = \frac{v_{13}^{(3)}}{\bar{v}_{13}^{(4)}} = \frac{1}{0.9452\,\frac{mL^3}{EI}} = 1.058\,\frac{EI}{mL^3}$$

$$\therefore \omega_1 = 1.029\sqrt{\frac{EI}{mL^3}}$$

$$\{\hat{v}_1\} = \begin{Bmatrix} -0.485 \\ -0.551 \\ 1 \end{Bmatrix}, \quad \omega_1 = 1.029\sqrt{\frac{EI}{mL^3}}$$

Problem 14-1



$$\ldots \quad \underline{k} = k \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\underline{m} = m\,\underline{I}$$

$$\underline{\Psi} = \begin{bmatrix} 1 & 1 \\ 0.25 & 0.56 \\ 0.50 & 0.25 \\ 0.25 & 0.06 \end{bmatrix} \qquad \therefore \quad \underline{k}^* = \underline{\Psi}^T \underline{k}\, \underline{\Psi} = k \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.3294 \end{bmatrix}$$

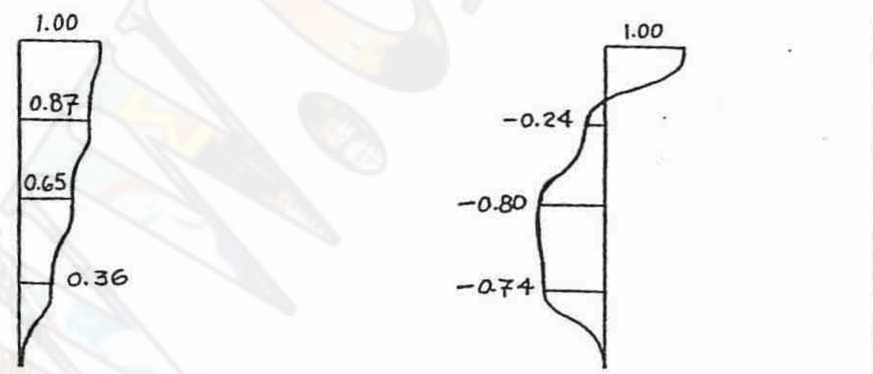
$$\underline{m}^* = \underline{\Psi}^T \underline{m}\, \underline{\Psi} = m \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.3294 \end{bmatrix}$$

From the discretized eigenvalue problem $[\underline{k}^* - \omega^2 \underline{m}^*]\,\hat{\underline{Z}} = \underline{0}$:

$$\underline{\Phi}_Z = \begin{bmatrix} 1 & 1 \\ -0.37668 & -1.23751 \end{bmatrix} \qquad \underline{\omega} = \sqrt{\frac{k}{m}} \begin{Bmatrix} 0.348 \\ 1.034 \end{Bmatrix}$$

then

$$\underline{\Phi} = \underline{\Psi}\, \underline{\Phi}_Z = \begin{bmatrix} 0.623 & -0.238 \\ 0.539 & 0.0570 \\ 0.406 & 0.1906 \\ 0.227 & 0.1757 \end{bmatrix}$$



► $\omega_1 = 0.348\sqrt{k/m}$ $\qquad \omega_2 = 1.034\sqrt{k/m}$

Problem 14-2

$$\underline{\Psi}^{(1)} = \underline{\tilde{f}}\,\underline{m}\,\underline{\Psi}^{(0)} = \underline{k}^{-1}\underline{m}\,\underline{\Psi}^{(0)}; \quad \underline{k}\,,\,\underline{m}\,,\,\underline{\Psi}^{(0)} = \underline{\Psi} \text{ from P14-1}$$

$$\therefore \quad \underline{\Psi}^{(1)} = \frac{m}{k}\begin{bmatrix} 7.50 & 6.24 \\ 6.50 & 5.24 \\ 4.75 & 3.68 \\ 2.50 & 1.87 \end{bmatrix}$$

Problem follows as P14-1, but using

$$\underline{k}^* = \left[\underline{\Psi}^{(1)}\right]^T \underline{m}\,\underline{\Psi}^{(0)}$$

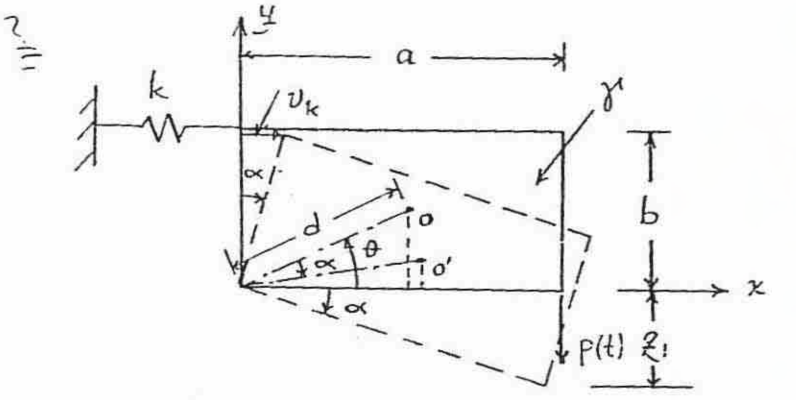
$$\underline{m}^* = \left[\underline{\Psi}^{(1)}\right]^T \underline{m}\,\underline{\Psi}^{(1)}$$

in $[k^* - \omega^2 m^*]\hat{\underline{z}} = \underline{0} \longrightarrow \underline{\Phi}_z, \underline{\omega}$

then $\underline{\Phi} = \underline{\Psi}^{(0)}\,\underline{\Phi}_z$

$$\blacktriangleright\ \underline{\Phi} = \begin{bmatrix} 0.635 & -0.2360 \\ 0.545 & 0.0576 \\ 0.409 & 0.1909 \\ 0.228 & 0.1758 \end{bmatrix}, \qquad \underline{\omega} = \sqrt{\frac{k}{m}}\begin{Bmatrix} 0.347 \\ 1.005 \end{Bmatrix}$$

Problem 16-1



$$d\cos\theta = \frac{a}{2}$$

$$d\sin\theta = \frac{b}{2}$$

$$\sin\alpha = \frac{q_1}{a} \longrightarrow \alpha = \arcsin\frac{q_1}{a}$$

$$\dot{\alpha} = \frac{1}{\sqrt{1-\left(\frac{q_1}{a}\right)^2}}\,\frac{\dot{q}_1}{a}$$

eq. (16-15): $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} + \dfrac{\partial V}{\partial q_i} = Q_i$

then: $T = \frac{1}{2} m \dot{v}_o^2 + \frac{1}{2} I_o \dot{\alpha}^2$

$V = \frac{1}{2} k\, v_k^2$

but,

$$x_o = d\cos(\theta-\alpha) = d\cos\theta\cos\alpha + d\sin\theta\sin\alpha$$

$$x_o = \frac{a}{2}\cos\alpha + \frac{b}{2}\sin\alpha$$

$$\dot{x}_o = -\frac{a}{2}\sin\alpha\,\dot{\alpha} + \frac{b}{2}\cos\alpha\,\dot{\alpha}$$

$$y_o = d\sin(\theta-\alpha) = d\sin\theta\cos\alpha - d\cos\theta\sin\alpha$$

$$y_o = \frac{b}{2}\cos\alpha - \frac{a}{2}\sin\alpha$$

$$\dot{y}_o = \frac{b}{2}\sin\alpha\,\dot{\alpha} + \frac{a}{2}\cos\alpha\,\dot{\alpha}$$

$$\therefore\ \dot{v}_o^2 = \dot{x}_o^2 + \dot{y}_o^2 = \left(\frac{a}{2}\right)^2\sin^2\alpha\,\dot{\alpha}^2 + \left(\frac{b}{2}\right)^2\cos^2\alpha\,\dot{\alpha}^2 - 2\,\frac{ab}{4}\sin\alpha\cos\alpha\,\dot{\alpha}^2$$
$$+\left(\frac{b}{2}\right)^2\sin^2\alpha\,\dot{\alpha}^2 + \left(\frac{a}{2}\right)^2\cos^2\alpha\,\dot{\alpha}^2 + 2\,\frac{ab}{4}\sin\alpha\cos\alpha\,\dot{\alpha}^2$$

$$\dot{v}_o^2 = \frac{a^2+b^2}{4}\,\dot{\alpha}^2\left(\sin^2\alpha + \cos^2\alpha\right) = \frac{a^2+b^2}{4}\,\dot{\alpha}^2$$

$$m = \gamma ab$$

$$I_o = m\,\frac{a^2+b^2}{12} = \gamma ab\,\frac{a^2+b^2}{12}$$

$$\therefore\ T = \frac{1}{2}\gamma ab\left(\frac{a^2+b^2}{4}\,\dot{\alpha}^2 + \frac{a^2+b^2}{12}\,\dot{\alpha}^2\right) = \frac{1}{6}\gamma ab\left(a^2+b^2\right)\dot{\alpha}^2$$

(continued on following page)

Problem 16-1 (con'd)

and $v_k = b\sin\alpha \longrightarrow \qquad V = \frac{1}{2} k b^2 \sin^2\alpha$

$$\therefore \quad T = \frac{1}{6}\gamma ab(a^2+b^2)\frac{(\dot{q}_1/a)^2}{1-\left(\frac{q_1}{a}\right)^2} \longrightarrow \frac{\partial T}{\partial \dot{q}_1} = \frac{1}{3}\gamma ab(a^2+b^2)\frac{\dot{q}_1/a^2}{1-\left(\frac{q_1}{a}\right)^2}$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{1}{3}\gamma ab(a^2+b^2)\frac{\dot{q}_1}{a^2-q_1^2}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{1}{3}\gamma ab(a^2+b^2)\frac{\ddot{q}_1(a^2-q_1^2)+\dot{q}_1 2q_1\dot{q}_1}{(a^2-q_1^2)^2}$$

$$\frac{\partial T}{\partial q_1} = \frac{1}{6}\gamma ab(a^2+b^2)\dot{q}_1^2\frac{2q_1}{(a^2-q_1^2)^2}$$

$$V = \frac{1}{2}kb^2\left(\frac{q_1}{a}\right)^2 \longrightarrow \qquad \frac{\partial V}{\partial q_1} = k\left(\frac{b}{a}\right)^2 q_1$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = \frac{1}{3}\gamma ab(a^2+b^2)\frac{\ddot{q}_1(a^2-q_1^2)+2q_1\dot{q}_1^2-q_1\dot{q}_1^2}{(a^2-q_1^2)^2} + k\left(\frac{b}{a}\right)^2 q_1$$

since $Q_1 = p(t)$

eq. of motion: $$\frac{1}{3}\gamma ab(a^2+b^2)\frac{(a^2-q_1^2)\ddot{q}_1+q_1\dot{q}_1^2}{(a^2-q_1^2)^2} + k\left(\frac{b}{a}\right)^2 q_1 = p(t)$$

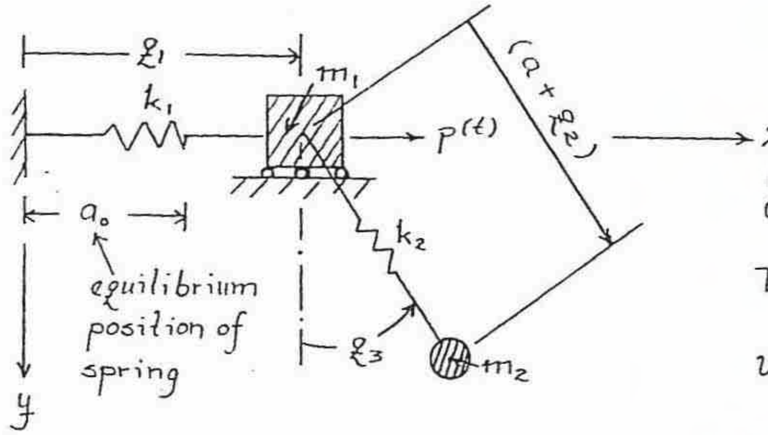
if q_1 small: $q_1^2 \simeq 0$, $\dot{q}_1^2 \simeq 0$

$$\therefore \frac{1}{3}\gamma ab(a^2+b^2)\frac{\ddot{q}_1}{a^2} + k\left(\frac{b}{a}\right)^2 q_1 = p(t)$$

$$\blacktriangleright \frac{1}{3}\gamma ab(a^2+b^2)\frac{(a^2-q_1^2)\ddot{q}_1+q_1\dot{q}_1^2}{(a^2-q_1^2)^2} + k\left(\frac{b}{a}\right)^2 q_1 = p(t)$$

q_1 small: $$\frac{1}{3}\gamma\frac{b}{a}(a^2+b^2)\ddot{q}_1 + k\left(\frac{b}{a}\right)^2 q_1 = p(t)$$

Problem 16-2



$a,\ a_0$: spring lengths

$$T = \frac{1}{2} m_1 \dot{v}_1^2 + \frac{1}{2} m_2 \dot{v}_2^2$$

$$v_1 = x_1 = q_1 \longrightarrow \dot{v}_1 = \dot{q}_1$$

$$x_2 = q_1 + (a+q_2)\sin q_3 \longrightarrow \dot{x}_2 = \dot{q}_1 + \dot{q}_2 \sin q_3 + (a+q_2)\cos q_3 \dot{q}_3$$

$$y_2 = (a+q_2)\cos q_3 \longrightarrow \dot{y}_2 = \dot{q}_2 \cos q_3 - (a+q_2)\sin q_3 \dot{q}_3$$

$$\therefore \dot{v}_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = \dot{q}_1^2 + \dot{q}_2^2 \sin^2 q_3 + (a+q_2)^2 \cos^2 q_3 \dot{q}_3^2 + 2\dot{q}_2 \sin q_3 (a+q_2)\cos q_3 \dot{q}_3$$
$$+ \dot{q}_2^2 \cos^2 q_3 + (a+q_2)^2 \sin^2 q_3 \dot{q}_3^2 - 2\dot{q}_2 \sin q_3 (a+q_2)\cos q_3 \dot{q}_3 + 2\dot{q}_1 \dot{q}_2 \sin q_3$$
$$+ 2\dot{q}_1 (a+q_2)\cos q_3 \dot{q}_3$$

$$\dot{v}_2^2 = \dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 \sin q_3 + 2\dot{q}_1 (a+q_2)\cos q_3 \dot{q}_3 + \dot{q}_2^2 + (a+q_2)^2 \dot{q}_3^2$$

$$\therefore T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \left[\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 \sin q_3 + 2\dot{q}_1 (a+q_2)\cos q_3 \dot{q}_3 + \dot{q}_2^2 + (a+q_2)^2 \dot{q}_3^2 \right]$$

$$T = \frac{1}{2}(m_1+m_2)\dot{q}_1^2 + \frac{1}{2} m_2 \left[2\dot{q}_1\dot{q}_2 \sin q_3 + 2\dot{q}_1\dot{q}_3 (a+q_2)\cos q_3 + \dot{q}_2^2 + (a+q_2)^2 \dot{q}_3^2 \right]$$

then,

$$\frac{\partial T}{\partial \dot{q}_1} = (m_1+m_2)\dot{q}_1 + m_2 \dot{q}_2 \sin q_3 + m_2 (a+q_2)\cos q_3 \dot{q}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1 + m_2 \ddot{q}_1 + m_2 \Big[\ddot{q}_2 \sin q_3 + \dot{q}_2 \cos q_3 \dot{q}_3 + \dot{q}_2 \cos q_3 \dot{q}_3 - (a+q_2)\sin q_3 \dot{q}_3^2$$
$$+ (a+q_2)\cos q_3 \ddot{q}_3 \Big]$$

$$\frac{\partial T}{\partial \dot{q}_2} = m_2 (\dot{q}_1 \sin q_3 + \dot{q}_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2 (\ddot{q}_1 \sin q_3 + \dot{q}_1 \cos q_3 \dot{q}_3 + \ddot{q}_2)$$

(continued on following page)

Problem 16-2 (con'd)

$$\frac{\partial T}{\partial \dot{q}_3} = m_2\left[\dot{q}_1(a+q_2)\cos q_3 + (a+q_2)^2\dot{q}_3\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_2\Big[\ddot{q}_1(a+q_2)\cos q_3 + \dot{q}_1\dot{q}_2\cos q_3 - \dot{q}_1(a+q_2)\sin q_3\dot{q}_3 + 2(a+q_2)\dot{q}_2\dot{q}_3 + (a+q_2)^2\ddot{q}_3\Big]$$

and

$$\frac{\partial T}{\partial q_1} = 0$$

$$\frac{\partial T}{\partial q_2} = m_2\left[\dot{q}_1\cos q_3\dot{q}_3 + (a+q_2)\dot{q}_3^2\right]$$

$$\frac{\partial T}{\partial q_3} = m_2\left[\dot{q}_1\dot{q}_2\cos q_3 - \dot{q}_1(a+q_2)\sin q_3\dot{q}_3\right]$$

$$V = \frac{1}{2}k_1 v_{k_1}^2 + \frac{1}{2}k_2 v_{k_2}^2 + m_2 g h_2$$

$$v_{k_1} = x_1 = q_1 - a_0$$

$$v_{k_2} = q_2$$

$$h_2 = -y_2 = -(a+q_2)\cos q_3$$

$$\therefore V = \frac{1}{2}k_1(q_1-a_0)^2 + \frac{1}{2}k_2 q_2^2 - m_2 g(a+q_2)\cos q_3$$

then

$$\frac{\partial V}{\partial q_1} = k_1(q_1 - a_0)$$

$$\frac{\partial V}{\partial q_2} = k_2 q_2 - m_2 g\cos q_3$$

$$\frac{\partial V}{\partial q_3} = m_2 g(a+q_2)\sin q_3$$

$Q_1 = p(t)$ and $Q_2 = Q_3 = 0$

$\therefore$ applying $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} + \dfrac{\partial V}{\partial q_i} = Q_i$ eq. (16-15)

(continued on following page)

Problem 16-2 (con'd)

$$m_1 \ddot{q}_1 + m_2\left\{\ddot{q}_1 + \left[(a+q_2)\ddot{q}_3 + 2\dot{q}_2\dot{q}_3\right]\cos q_3 + \left[\ddot{q}_2 - (a+q_2)\dot{q}_3^2\right]\sin q_3\right\} + k_1(q_1 - a_0) = p(t)$$

$$m_2\left(\ddot{q}_2 + \dot{q}_1\dot{q}_3\cos q_3 + \ddot{q}_1 \sin q_3\right) - m_2\left[(a+q_2)\dot{q}_3^2 - \dot{q}_1\dot{q}_3\cos q_3\right] + k_2 q_2 - m_2 g\cos q_3 = 0$$

$$m_2\left\{(a+q_2)^2\ddot{q}_3 + 2(a+q_2)\dot{q}_2\dot{q}_3 + \left[(a+q_2)\ddot{q}_1 + \dot{q}_1\dot{q}_2\right]\cos q_3 - (a+q_2)\dot{q}_1\dot{q}_3\sin q_3\right\}$$

$$- m_2\left[\dot{q}_1\dot{q}_2\cos q_3 - (a+q_2)\dot{q}_1\dot{q}_3\sin q_3\right] + m_2 g(a+q_2)\sin q_3 = 0$$

$$\therefore \quad (m_1+m_2)\ddot{q}_1 + m_2\left\{\left[(a+q_2)\ddot{q}_3 + 2\dot{q}_2\dot{q}_3\right]\cos q_3 + \left[\ddot{q}_2 - (a+q_2)\dot{q}_3^2\right]\sin q_3\right\}$$

$$+ k_1(q_1 - a_0) = p(t)$$

$$m_2\ddot{q}_2 - m_2\left[(a+q_2)\dot{q}_3^2 + \ddot{q}_1\sin q_3\right] + k_2 q_2 - m_2 g\cos q_3 = 0$$

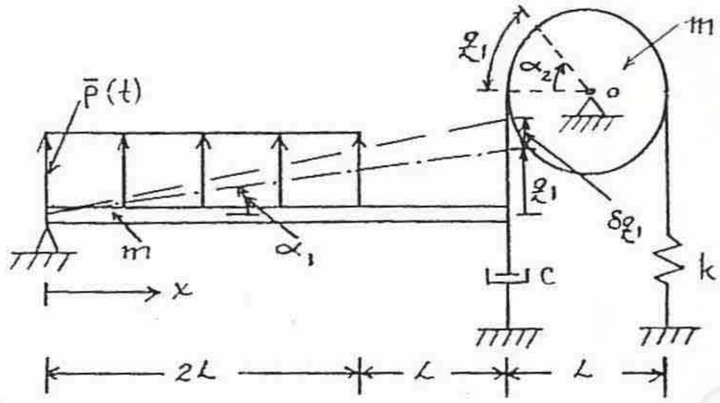
$$m_2(a+q_2)^2\ddot{q}_3 + m_2(a+q_2)\left(2\dot{q}_2\dot{q}_3 + \ddot{q}_1\cos q_3\right) + m_2 g(a+q_2)\sin q_3 = 0$$

if q_i small:

$$\left.\begin{aligned} &\cos q_i = 1, \quad \sin q_i \approx q_i \\ &q_i q_j \simeq 0 \\ &\dot{q}_i \dot{q}_j \simeq 0 \\ &\ddot{q}_i q_j \simeq 0 \end{aligned}\right\} \quad i, j = 1, 2, 3$$

$$\blacktriangleright \quad \begin{aligned} (m_1+m_2)\ddot{q}_1 + m_2 a\ddot{q}_3 + k_1(q_1 - a_0) &= p(t) \\ m_2\ddot{q}_2 \qquad + k_2 q_2 &= m_2 g \\ \ddot{q}_1 + \quad a\ddot{q}_3 + g\, q_3 &= 0 \end{aligned}$$

Problem 16-3



$$\sin \alpha_1 = \frac{q_1}{3L}$$

$$\therefore \quad \alpha_1 = \arcsin \frac{q_1}{3L}$$

$$\dot{\alpha}_1 = \frac{\dot{q}_1/(3L)}{\sqrt{1-\left(\frac{q_1}{3L}\right)^2}}$$

$$q_1 = \alpha_2 \frac{L}{2} \longrightarrow \alpha_2 = \frac{2q_1}{L}$$

$$\therefore \quad \dot{\alpha}_2 = \frac{2\dot{q}_1}{L}$$

$$* \longrightarrow T = \frac{1}{2} I_A \dot{\alpha}_1^2 + \frac{1}{2} I_o \dot{\alpha}_2^2 \ , \quad I_A = m\frac{(3L)^2}{12} + m\left(\frac{3L}{2}\right)^2 = 3mL^2$$

$$I_o = m\,\frac{L^2+L^2}{16} = \frac{1}{8} mL^2$$

$$\therefore \ T = \frac{3}{2} mL^2 \, \frac{(\dot{q}_1/3L)^2}{1-\left(\frac{q_1}{3L}\right)^2} + \frac{1}{16} mL^2 \left(\frac{2\dot{q}_1}{L}\right)^2$$

$$T = \frac{m}{12}\left[\frac{2}{1-(q_1/3L)^2} + 3\right]\dot{q}_1^2 = \frac{m}{12}\,\frac{5-3(q_1/3L)^2}{1-(q_1/3L)^2}\,\dot{q}_1^2$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{m}{6}\,\frac{5-3(q_1/3L)^2}{1-(q_1/3L)^2}\,\dot{q}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{m}{6}\,\frac{\left\{\left[5-3(q_1/3L)^2\right]\ddot{q}_1 - (2/3L^2)q_1\dot{q}_1^2\right\}\left[1-(q_1/3L)^2\right] + \left[5-3(q_1/3L)^2\right]\dot{q}_1(2/9L^2)q_1\dot{q}_1^2}{\left[1-(q_1/3L)^2\right]^2}$$

and
$$\frac{\partial T}{\partial q_1} = \frac{m}{12}\,\dot{q}_1^2\,\frac{-(2/3L^2)\,q_1\left[1-(q_1/3L)^2\right] + (2/9L^2)\,q_1\left[5-3(q_1/3L)^2\right]}{\left[1-(q_1/3L)^2\right]^2}$$

$$* \longrightarrow V = \frac{1}{2} k v_k^2 = \frac{1}{2} k q_1^2 \longrightarrow \frac{\partial V}{\partial q_1} = k q_1$$

$$* \longrightarrow \text{Impose} \quad \delta q_1 \longrightarrow \delta W_{nc}(x) = \bar{p}(t)\frac{\delta q_1}{3L} x - c\dot{q}_1 \delta q_1$$

$$\therefore \quad \delta W_{nc} = \left[\frac{\bar{p}(t)}{3L}\int_0^{2L} x\,dx - c\dot{q}_1\right]\delta q_1$$

$$\delta W_{nc} = \left[\frac{\bar{p}(t)}{3L}\,\frac{1}{2}\,4L^2 - c\dot{q}_1\right]\delta q_1$$

(continued on following page)

Problem 16-3 (con'd)

$$\delta W_{nc} = \left[\frac{2}{3}\,\bar{p}(t)L - c\dot{Z}_1\right]\delta Z_1$$

Comparing with eq. (16-11c): $\delta W_{nc} = Q_1\,\delta Z_1 + Q_2\,\delta Z_2 + \cdots + Q_N\,\delta Z_N$

$$\therefore\ Q_1 = \frac{2}{3}\,\bar{p}(t)L - c\dot{Z}_1$$

applying eq. (16-15): $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{Z}_1}\right) - \dfrac{\partial T}{\partial Z_1} + \dfrac{\partial V}{\partial Z_1} = Q_1$

$$\frac{m}{6}\,\frac{[5-3(Z_1/3L)^2][1-(Z_1/3L)^2]\ddot{Z}_1 - (2/3L^2)[1-(Z_1/3L)^2]Z_1\dot{Z}_1^2 + [5-3(Z_1/3L)^2](2/9L^2)Z_1\dot{Z}_1^2}{[1-(Z_1/3L)^2]^2}$$

$$-\frac{m}{6}\,\frac{(1/9L^2)Z_1\dot{Z}_1^2[5-3(Z_1/3L)^2] - (1/3L^2)Z_1\dot{Z}_1^2[1-(Z_1/3L)^2]}{[1-(Z_1/3L)^2]^2} + kZ_1 = \frac{2}{3}\bar{p}(t)L - c\dot{Z}_1$$

$$\frac{m}{6}\,\frac{[5-3(Z_1/3L)^2][1-(Z_1/3L)^2]\ddot{Z}_1 - [1-(Z_1/3L)^2](Z_1\dot{Z}_1^2/3L^2) + [5-3(Z_1/3L)^2](Z_1\dot{Z}_1^2/9L^2)}{[1-(Z_1/3L)^2]^2}$$

$$+kZ_1 = \frac{2}{3}\bar{p}(t)L - c\dot{Z}_1$$

$$\frac{m}{6}\,\frac{[5-3(Z_1/3L)^2][1-(Z_1/3L)^2]\ddot{Z}_1 + (Z_1\dot{Z}_1^2/9L^2)2}{[1-(Z_1/3L)^2]^2} \qquad +kZ_1 = \frac{2}{3}\bar{p}(t)L - c\dot{Z}_1$$

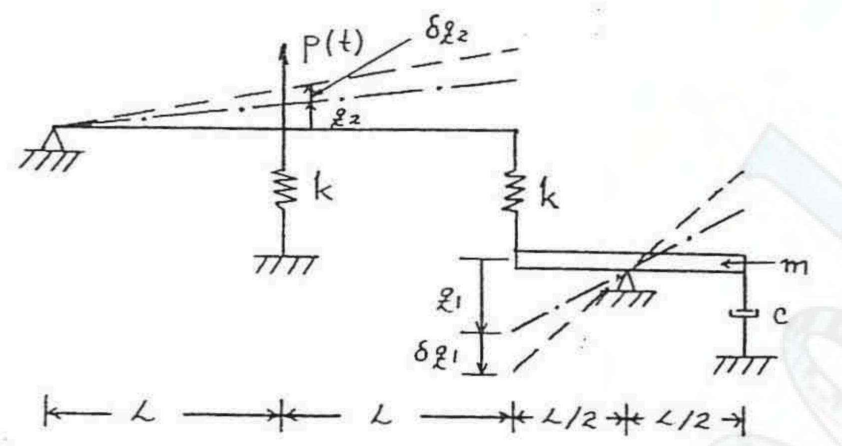
if Z_1 small: $Z_1^2,\ \dot{Z}_1^2 \approx 0$

then $\dfrac{5m}{6}\ddot{Z}_1 + kZ_1 = \dfrac{2}{3}\bar{p}(t)L - c\dot{Z}_1$

$$\blacktriangleright\ \frac{m}{4}\,\frac{[5-3(Z_1/3L)^2][1-(Z_1/3L)^2]\ddot{Z}_1 + (2/9L^2)Z_1\dot{Z}_1^2}{[1-(Z_1/3L)^2]^2} + \frac{3}{2}c\dot{Z}_1 + \frac{3}{2}kZ_1 = \bar{p}(t)L$$

if Z_1 small: $\qquad \dfrac{5}{4}m\ddot{Z}_1 + \dfrac{3}{2}c\dot{Z}_1 + \dfrac{3}{2}kZ_1 = \bar{p}(t)L$

Problem 16-4



$\longrightarrow T = \frac{1}{2} I_0 \dot{\alpha}^2$, but $I_0 = m\frac{L^2}{12}$

$$\sin\alpha = \frac{q_1}{L/2} \longrightarrow \alpha = \arcsin\frac{2q_1}{L}$$

$$\dot{\alpha} = \frac{2\dot{q}_1/L}{\sqrt{1-\left(\frac{2q_1}{L}\right)^2}}$$

$$\therefore T = \frac{1}{2}\frac{mL^2}{12}\frac{\frac{4\dot{q}_1^2}{L^2}}{1-\left(\frac{2q_1}{L}\right)^2} = \frac{m}{6}\frac{\dot{q}_1^2}{1-\left(\frac{2q_1}{L}\right)^2}$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{m}{3}\frac{\dot{q}_1}{1-\left(\frac{2q_1}{L}\right)^2} \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{m}{3}\frac{\ddot{q}_1\left[1-(2q_1/L)^2\right]+\dot{q}_1^2(4/L^2)2q_1}{\left[1-(2q_1/L)^2\right]^2}$$

$$\frac{\partial T}{\partial \dot{q}_2} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = 0$$

$$\frac{\partial T}{\partial q_1} = \frac{m}{6}\dot{q}_1^2\frac{2(4/L^2)q_1}{\left[1-(2q_1/L)^2\right]^2}, \qquad \frac{\partial T}{\partial q_2} = 0$$

$\longrightarrow V = \frac{1}{2}kv_2^2 + \frac{1}{2}kv_1^2$, but $v_1 = q_1 + 2q_2$

$$v_2 = q_2$$

$$\therefore V = \frac{1}{2}k\left[q_2^2 + (q_1+2q_2)^2\right]$$

$$\frac{\partial V}{\partial q_1} = \frac{1}{2}k\,2(q_1+2q_2) = k(q_1+2q_2), \quad \frac{\partial V}{\partial q_2} = \frac{1}{2}k\left[2q_2 + 2(q_1+2q_2)2\right] = k(2q_1+5q_2)$$

(continued on following page)

Problem 16-4 (con'd)

* $\rightarrow$ Imposing δq_1, δq_2: $\quad \delta W_{nc} = p(t)\,\delta q_2 - c\dot{q}_1\,\delta q_1$

comparing with eq. (16-11c): $\quad \delta W_{nc} = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \cdots + Q_N\,\delta q_N$

$$Q_1 = -c\dot{q}_1$$

$$Q_2 = p(t)$$

applying eq. (16-15): $\quad \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} + \dfrac{\partial V}{\partial q_i} = Q_i$

$$\frac{m}{3}\,\frac{\ddot{q}_1\left[1-(2q_1/L)^2\right] + \dot{q}_1^2(8/L^2)q_1}{\left[1-(2q_1/L)^2\right]^2} - \frac{m}{6}\dot{q}_1^2\,\frac{8q_1/L^2}{\left[1-(2q_1/L)^2\right]^2} + k(q_1+2q_2) = -c\dot{q}_1$$

$$k(2q_1+5q_2) = p(t)$$

$$\therefore \frac{m}{3}\,\frac{\left[1-(2q_1/L)^2\right]\ddot{q}_1 + 4(q_1\dot{q}_1^2/L^2)}{\left[1-(2q_1/L)^2\right]^2} + kq_1 + \frac{2}{5}\left[p(t) - 2kq_1\right] = -c\dot{q}_1$$

if q_1 small $\longrightarrow q_1^2\,;\ \dot{q}_1^2 \approx 0$

$$\therefore \frac{m}{3}\ddot{q}_1 + \frac{1}{5}kq_1 + \frac{2}{5}p(t) = -c\dot{q}_1$$

$$\blacktriangleright\ \frac{5}{6}m\,\frac{\left[1-(2q_1/L)^2\right]\ddot{q}_1 + 4(q_1\dot{q}_1^2/L^2)}{\left[1-(2q_1/L)^2\right]^2} + \frac{5}{2}c\dot{q}_1 + \frac{1}{2}kq_1 = -p(t)$$

if q_1 small: $\quad \dfrac{5}{6}m\ddot{q}_1 + \dfrac{5}{2}c\dot{q}_1 + \dfrac{1}{2}kq_1 = -p(t)$

Problem 16-5

$$v(x,t) = q_1(t)\left(\frac{x}{L}\right)^2 + q_2(t)\left(\frac{x}{L}\right)^3 + q_3(t)\left(\frac{x}{L}\right)^4 = \sum_{i=1}^{3} q_i(t)\left(\frac{x}{L}\right)^{i+1}$$

$$\therefore\ \psi_i(x) = \left(\frac{x}{L}\right)^{i+1}\ ,\ i = 1,2,3$$

eq. (16-22):
$$m_{ij} = \int m(x)\,\psi_i(x)\,\psi_j(x)\,dx = \int_0^L \bar{m}\left(\frac{x}{L}\right)^{i+1}\left(\frac{x}{L}\right)^{j+1} dx$$

$$m_{ij} = \bar{m}\int_0^L \left(\frac{x}{L}\right)^{i+j+2} dx = \bar{m}\,\frac{1}{i+j+3}\,L = \frac{\bar{m}L}{i+j+3}$$

First eq. above eq. (16-22):
$$T = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{N} m_{ij}\,\dot{q}_i\,\dot{q}_j = \frac{\bar{m}L}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\dot{q}_i\,\dot{q}_j}{i+j+3}$$

$$T = \bar{m}L\left(\frac{1}{5}\dot{q}_1^2 + \frac{1}{6}\dot{q}_1\dot{q}_2 + \frac{1}{7}\dot{q}_1\dot{q}_3 + \frac{1}{7}\dot{q}_2^2 + \frac{1}{8}\dot{q}_2\dot{q}_3 + \frac{1}{9}\dot{q}_3^2\right)$$

$$\frac{\partial T}{\partial \dot{q}_1} = \bar{m}L\left(\frac{2}{5}\dot{q}_1 + \frac{1}{6}\dot{q}_2 + \frac{1}{7}\dot{q}_3\right) \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \bar{m}L\left(\frac{2}{5}\ddot{q}_1 + \frac{1}{6}\ddot{q}_2 + \frac{1}{7}\ddot{q}_3\right)$$

$$\frac{\partial T}{\partial \dot{q}_2} = \bar{m}L\left(\frac{1}{6}\dot{q}_1 + \frac{2}{7}\dot{q}_2 + \frac{1}{8}\dot{q}_3\right) \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = \bar{m}L\left(\frac{1}{6}\ddot{q}_1 + \frac{2}{7}\ddot{q}_2 + \frac{1}{8}\ddot{q}_3\right)$$

$$\frac{\partial T}{\partial \dot{q}_3} = \bar{m}L\left(\frac{1}{7}\dot{q}_1 + \frac{1}{8}\dot{q}_2 + \frac{2}{9}\dot{q}_3\right) \longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = \bar{m}L\left(\frac{1}{7}\ddot{q}_1 + \frac{1}{8}\ddot{q}_2 + \frac{2}{9}\ddot{q}_3\right)$$

eq. (16-24):
$$k_{ij} = \int EI(x)\,\psi_i''(x)\,\psi_j''(x)\,dx = \int_0^L EI\,(i+1)\,i\,\frac{x^{(i-1)}}{L^{(i+1)}}\,(j+1)\,j\,\frac{x^{(j-1)}}{L^{(j+1)}}\,dx$$

$$k_{ij} = (i+1)(j+1)\,ij\,EI\,\frac{1}{L^{(i+j+2)}}\int_0^L x^{(i+j-2)}\,dx = \frac{(1+i)(1+j)}{i+j-1}\,ij\,\frac{L^{(i+j-1)}}{L^{(i+j+2)}}\,EI$$

$$k_{ij} = \frac{(1+i)(1+j)}{i+j-1}\,ij\,\frac{EI}{L^3}$$

eq. (10-42):
$$k_{Gij} = \int_0^L N(x)\,\psi_i'(x)\,\psi_j'(x)\,dx = \int_0^L N\,(i+1)\,\frac{x^i}{L^{(i+1)}}\,(j+1)\,\frac{x^j}{L^{(j+1)}}\,dx$$

$$k_{Gij} = (1+i)(1+j)\,\frac{N}{L^{(i+j+2)}}\int_0^L x^{(i+j)}\,dx = (1+i)(1+j)\,\frac{N}{L^{(i+j+2)}}\,\frac{L^{(i+j+1)}}{(i+j+1)}$$

$$k_{Gij} = \frac{(1+i)(1+j)}{(i+j+1)}\,\frac{N}{L}$$

(continued on following page)

Problem 16-5 (con'd)

since $\bar{k}_{ij} = k_{ij} - k_{Gij}$,

$$\bar{k}_{ij} = \frac{(1+i)(1+j)}{i+j-1}\, ij \frac{EI}{L^3} - \frac{(1+i)(1+j)}{i+j+1}\frac{N}{L}$$

from eq. (16-17):

$$V = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{N} k_{ij} q_i q_j \;,$$

$$\therefore\; V = \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\left[\frac{(1+i)(1+j)}{i+j-1}\, ij \frac{EI}{L^3} - \frac{(1+i)(1+j)}{i+j+1}\frac{N}{L}\right] q_i q_j$$

$$V = \left(4\frac{EI}{L^3} - \frac{4}{3}\frac{N}{L}\right) q_1^2 + \left(6\frac{EI}{L^3} - \frac{3}{2}\frac{N}{L}\right) q_1 q_2 + \left(8\frac{EI}{L^3} - \frac{8}{5}\frac{N}{L}\right) q_1 q_2$$
$$+ \left(12\frac{EI}{L^3} - \frac{9}{5}\frac{N}{L}\right) q_2^2 + \left(18\frac{EI}{L^3} - 2\frac{N}{L}\right) q_2 q_3 + \left(\frac{144}{5}\frac{EI}{L^3} - \frac{16}{7}\frac{N}{L}\right) q_3^2$$

since $v(0,t)=0$ and $\dot{v}(0,t)=0$, there is no necessity to impose restrictions, then

$$\frac{\partial V}{\partial q_1} = 8\left(\frac{EI}{L^3} - \frac{1}{3}\frac{N}{L}\right) q_1 + 3\left(2\frac{EI}{L^3} - \frac{1}{2}\frac{N}{L}\right) q_2 + 8\left(\frac{EI}{L^3} - \frac{1}{5}\frac{N}{L}\right) q_3$$

$$\frac{\partial V}{\partial q_2} = 3\left(2\frac{EI}{L^3} - \frac{1}{2}\frac{N}{L}\right) q_1 + 6\left(4\frac{EI}{L^3} - \frac{3}{5}\frac{N}{L}\right) q_2 + 2\left(9\frac{EI}{L^3} - \frac{N}{L}\right) q_3$$

$$\frac{\partial V}{\partial q_3} = 8\left(\frac{EI}{L^3} - \frac{1}{5}\frac{N}{L}\right) q_1 + 2\left(9\frac{EI}{L^3} - \frac{N}{L}\right) q_2 + 32\left(\frac{9}{5}\frac{EI}{L^3} - \frac{1}{7}\frac{N}{L}\right) q_3$$

eq. (16-30): $\quad p_i = \int p(x,t)\,\psi_i(x)\,dx = p(t)(1)^{(i+1)} = p(t) \quad , \quad i=1,2,3$

eq. (16-31): $\quad c_{ij} = a_1 \int EI(x)\,\psi_i''(x)\,\psi_j''(x)\,dx = 0 \quad , \quad i=1,2,3$

eq. (16-32): $\quad Q_i = p_i - \sum_{j=1}^{N} c_{ij}\dot{q}_j = p(t) \quad , \quad i=1,2,3$

eq. (16-18): $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = Q_i \quad , \quad i=1,2,3$

(continued on following page)

Problem 16-5 (con'd)

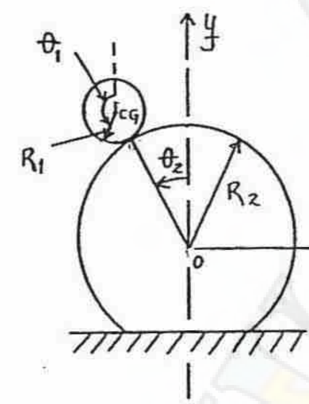
$$\bar{m}L\left(\frac{2}{5}\ddot{q}_1+\frac{1}{6}\ddot{q}_2+\frac{1}{7}\ddot{q}_3\right)+\frac{EI}{L^3}\left(8q_1+6q_2+8q_3\right)-\frac{N}{L}\left(\frac{8}{3}q_1+\frac{3}{2}q_2+\frac{8}{5}q_3\right)=p(t)$$

$$\bar{m}L\left(\frac{1}{6}\ddot{q}_1+\frac{2}{7}\ddot{q}_2+\frac{1}{8}\ddot{q}_3\right)+\frac{EI}{L^3}\left(6q_1+24q_2+18q_3\right)-\frac{N}{L}\left(\frac{3}{2}q_1+\frac{18}{5}q_2+2q_3\right)=p(t)$$

$$\bar{m}L\left(\frac{1}{7}\ddot{q}_1+\frac{1}{8}\ddot{q}_2+\frac{2}{9}\ddot{q}_3\right)+\frac{EI}{L^3}\left(8q_1+18q_2+288q_3\right)-\frac{N}{L}\left(\frac{8}{5}q_1+2q_2+\frac{32}{7}q_3\right)=p(t)$$

$$\blacktriangleright\ \frac{\bar{m}L}{2520}\begin{bmatrix}1008 & 420 & 360\\ 420 & 720 & 315\\ 360 & 315 & 560\end{bmatrix}\begin{Bmatrix}\ddot{q}_1(t)\\ \ddot{q}_2(t)\\ \ddot{q}_3(t)\end{Bmatrix}+\left[\frac{2EI}{L^3}\begin{bmatrix}4 & 3 & 4\\ 3 & 12 & 9\\ 4 & 9 & 144\end{bmatrix}-\frac{N}{210L}\begin{bmatrix}560 & 315 & 336\\ 315 & 756 & 420\\ 336 & 420 & 960\end{bmatrix}\right]\begin{Bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{Bmatrix}=\begin{Bmatrix}1\\1\\1\end{Bmatrix}p(t)$$

Problem 16-6



(a) $R_1\theta_1 = R_2\theta_2$

$$\blacktriangleright\ \theta_1=\frac{R_2}{R_1}\theta_2$$

(b) $* \rightarrow T=\frac{1}{2}m_1\dot{v}^2+\frac{1}{2}I_{01}(\dot{\theta}_1+\dot{\theta}_2)^2$

$x_1=-(R_1+R_2)\sin\theta_2 \rightarrow \dot{x}_1=-(R_1+R_2)\cos\theta_2\,\dot{\theta}_2$

$y_1=(R_1+R_2)\cos\theta_2 \rightarrow \dot{y}_1=-(R_1+R_2)\sin\theta_2\,\dot{\theta}_2$

Ball subjected to two rotations

- CG rotates in θ_2 respect to 0
- Ball rotates in θ_1 recpect to CG

$\therefore\ \dot{v}^2=(R_1+R_2)^2\,\dot{\theta}_2^2$

$I_{01}=\frac{2}{5}m_1R_1^2$

(continued on following page)

Problem 16-6 (con'd)

then $T = \frac{1}{2} m_1 (R_1 + R_2)^2 \dot{\theta}_2^2 + \frac{1}{2}\frac{2}{5} m_1 R_1^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$

$$T = \frac{1}{2} m_1 (R_1 + R_2)^2 \dot{\theta}_2^2 + \frac{1}{2}\frac{2}{5} m_1 R_1^2 \left(\frac{R_2}{R_1}\dot{\theta}_2 + \dot{\theta}_2\right)^2 = \frac{1}{2}\frac{7}{5} m_1 (R_1+R_2)^2 \dot{\theta}_2^2$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{7}{5} m_1 (R_1 + R_2)^2 \dot{\theta}_2, \qquad \frac{\partial T}{\partial \theta_2} = 0$$

$\ast\longrightarrow$ $V = m_1 g h_1, \quad h_1 = y_1 = (R_1 + R_2)\cos\theta_2$

then $V = m_1 g (R_1 + R_2)\cos\theta_2$

$\therefore \frac{\partial V}{\partial \theta_2} = -m_1 g (R_1 + R_2)\sin\theta_2$

$\ast\longrightarrow$ $Q_2 = 0$

eq. (16-15): $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$

$$\frac{7}{5} m_1 (R_1 + R_2)^2 \ddot{\theta}_2 - m_1 g (R_1 + R_2)\sin\theta_2 = 0$$

$$\boxed{\ddot{\theta}_2 - \frac{5g}{7(R_1+R_2)}\sin\theta_2 = 0}$$

(c)

$\ast\longrightarrow$ $T = \frac{1}{2} m_1 (R_1 + R_2)^2 \dot{\theta}_2^2 + \frac{1}{2}\frac{2}{5} m_1 R_1^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$

$\therefore \frac{\partial T}{\partial \dot{\theta}_1} = \frac{2}{5} m_1 R_1^2 (\dot{\theta}_1 + \dot{\theta}_2), \qquad \frac{\partial T}{\partial \theta_1} = 0$

$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 (R_1 + R_2)^2 \dot{\theta}_2 + \frac{2}{5} m_1 R_1^2 (\dot{\theta}_1 + \dot{\theta}_2), \qquad \frac{\partial T}{\partial \theta_2} = 0$

$\ast\longrightarrow$ eq. (16-39): $\overline{V} = V(q_1, q_2, \cdots, q_c) - (\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m)$

$$V = m_1 g (R_1 + R_2)\cos\theta_2$$

$$f_1 = R_1\theta_1 - R_2\theta_2 = 0$$

$$\therefore \overline{V} = mg (R_1 + R_2)\cos\theta_2 - \lambda_1 (R_1\theta_1 - R_2\theta_2)$$

(continued on following page)

Problem 16-6 (con'd)

then $\dfrac{\partial \bar{V}}{\partial \theta_1} = -\lambda_1 R_1$

$$\frac{\partial \bar{V}}{\partial \theta_2} = -mg(R_1+R_2)\sin\theta_2 + \lambda_1 R_2$$

$\ast \longrightarrow \quad Q_1 = Q_2 = 0$

eq. (16-40) : $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_i}\right) - \dfrac{\partial T}{\partial q_i} + \dfrac{\partial \bar{V}}{\partial q_i} = Q_i \quad , \; i = 1, 2, \ldots, c$

$$\frac{2}{5} m_1 R_1^2 (\ddot{\theta}_1 + \ddot{\theta}_2) - \lambda_1 R_1 = 0$$

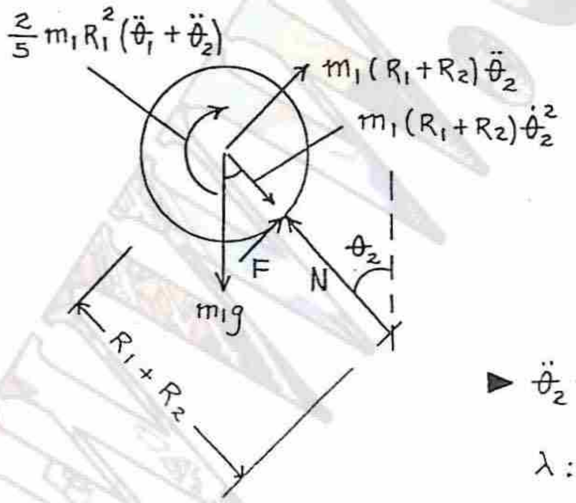
$$m_1 (R_1+R_2)^2 \ddot{\theta}_2 + \frac{2}{5} m_1 R_1^2 (\ddot{\theta}_1 + \ddot{\theta}_2) - m_1 g (R_1+R_2)\sin\theta_2 + \lambda_1 R_2 = 0$$

$$\frac{2}{5}\cancel{m_1} R_1^{\cancel{2}} (\ddot{\theta}_1 + \ddot{\theta}_2) R_2 + \cancel{m_1}(R_1+R_2)^2 \ddot{\theta}_2 \cancel{R_1} + \frac{2}{5}\cancel{m_1} R_1^2 (\ddot{\theta}_1 + \ddot{\theta}_2)\cancel{R_1}$$
$$- \cancel{m_1} g (R_1+R_2)\sin\theta_2 \cancel{R_1} = 0$$

since $\theta_1 = \dfrac{R_2}{R_1}\theta_2$, $\left[\dfrac{2}{5}R_1\left(\dfrac{R_2}{R_1}+1\right)R_2 + (R_1+R_2)^2 + \dfrac{2}{5}R_1^2\left(\dfrac{R_2}{R_1}+1\right)\right]\ddot{\theta}_2 - g(R_1+R_2)\sin\theta_2 = 0$

$$\left[\frac{2}{5}(R_2^2 + 2R_1R_2 + R_1^2) + (R_1+R_2)^2\right]\ddot{\theta}_2 - g(R_1+R_2)\sin\theta_2 = 0$$

$$\therefore \quad \ddot{\theta}_2 - \frac{5g}{7(R_1+R_2)}\sin\theta_2 = 0$$



$$F \cdot R_1 = \frac{2}{5} m_1 R_1^2 (\ddot{\theta}_1 + \ddot{\theta}_2) = \lambda R_1$$

$$\therefore \; F = \lambda$$

$$\blacktriangleright \; \ddot{\theta}_2 - \frac{5g}{7(R_1+R_2)}\sin\theta_2 = 0$$

λ : friction force between ball and cylinder

(continued on following page)

Problem 16-6 (con'd)

(d)

From above sketch: $\vec{F} = m\vec{a} \longrightarrow -m_1(R_1+R_2)\dot{\theta}_2^2 = N - m_1 g \cos\theta_2$

Ball leaves cylinder when $N = 0$, i.e., $\dot{\theta}_2^2 - \dfrac{g}{R_1+R_2}\cos\theta_2 = 0$

From eq. of movement: $$\ddot{\theta}_2 = \frac{5g}{7(R_1+R_2)}\sin\theta_2$$

$$\ddot{\theta}_2\dot{\theta}_2 = \frac{5g}{7(R_1+R_2)}\sin\theta_2\dot{\theta}_2$$

$$\therefore \quad \frac{1}{2}\dot{\theta}_2^2 = \frac{-5g}{7(R_1+R_2)}\cos\theta_2 + A$$

@ $t=0$, $\theta_2, \dot{\theta}_2 = 0$. $$0 = \frac{-5g}{7(R_1+R_2)} + A$$

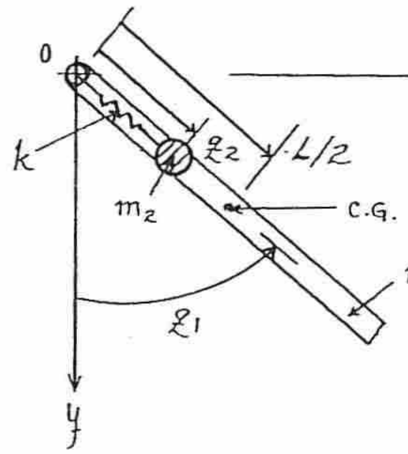
$$\therefore \quad \dot{\theta}_2^2 = \frac{10g}{7(R_1+R_2)}(1-\cos\theta_2)$$

Hence $$\frac{10g}{7(R_1+R_2)}(1-\cos\theta_2) - \frac{g}{R_1+R_2}\cos\theta_2 = 0$$

$$\cos\theta_2 = \frac{10}{17} \longrightarrow \theta_2 = \arccos\frac{10}{17}$$

$$\blacktriangleright\ \theta_2 = \arccos\frac{10}{17} = 54.0^\circ$$

Problem 16-7



- Kinetic energy: $T = \frac{1}{2}(I_o)_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{v}_2^2$

$$(I_o)_1 = I_{CG} + m_1\left(\frac{L}{2}\right)^2 = m_1 \frac{L^2}{12} + m_1 \frac{L^2}{4} = \frac{1}{3} m_1 L^2$$

$$x_2 = q_2 \sin q_1 \longrightarrow \dot{x}_2 = \dot{q}_2 \sin q_1 + q_2 \dot{q}_1 \cos q_1$$

$$y_2 = q_2 \cos q_1 \longrightarrow \dot{y}_2 = \dot{q}_2 \cos q_1 - q_2 \dot{q}_1 \sin q_1$$

$$\therefore \quad \dot{v}_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = \dot{q}_2^2 + q_2^2 \dot{q}_1^2$$

$$\therefore \quad T = \frac{1}{6} m_1 L^2 \dot{q}_1^2 + \frac{1}{2} m_2 \left(\dot{q}_2^2 + q_2^2 \dot{q}_1^2\right)$$

- Potencial energy: $V = m_1 g h_1 + m_2 g h_2 + \frac{1}{2} k q_2^2$ (note : spring length = 0)

$$h_1 = -h_{CG} = -\frac{L}{2} \cos q_1$$

$$h_2 = -y_2 = -q_2 \cos q_1$$

$$\therefore \quad V = -m_1 g \frac{L}{2} \cos q_1 - m_2 g q_2 \cos q_1 + \frac{1}{2} k q_2^2$$

- Generalized forcing functions: $Q_1 = Q_2 = 0$

(continued on following page)

Problem 16-7 (con'd)

- Equilibrium equations : $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$ for $i = 1, 2$

$$\left(\frac{1}{3} m_1 L^2 + m_2 q_2^2\right)\ddot{q}_1 + 2 m_2 q_2 \dot{q}_2 \dot{q}_1 + \left(\frac{L}{2} m_1 + q_2 m_2\right) g \sin q_1 = 0$$

$$m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 - m_2 g \cos q_1 + k q_2 = 0$$

Problem 16-8

For small amplitude oscillations:

$q_i q_j ,\ \dot{q}_i \dot{q}_j \approx 0 ,\quad \sin q_i \approx q_i ,\quad \cos q_i \approx 1 \qquad (i, j = 1, 2)$

$$\frac{L}{3} \ddot{q}_1 + \frac{1}{2} q_1 = 0$$

$$m_2 \ddot{q}_2 + k q_2 = m_2 g$$